A Solutions

1. (a) We have

$$\begin{aligned}
\int \sec^4\left(\frac{x}{2}\right) dx &= \int \sec^2\left(\frac{x}{2}\right)\sec^2\left(\frac{x}{2}\right) dx = \int \left[1+\tan^2\left(\frac{x}{2}\right)\right]\sec^2\left(\frac{x}{2}\right) dx \\
&= \int 2(1+u^2)\,du, \qquad \text{(substituting } u = \tan(x/2)\text{)} \\
&= 2u + \frac{2}{3}u^3 + C = 2\tan\left(\frac{x}{2}\right) + \frac{2}{3}\tan^3\left(\frac{x}{2}\right) + C.
\end{aligned}$$

(b) This is one for which we use integration by parts:

$$\begin{aligned}
\int x\sin(3x)\,dx &= -\frac{1}{3}x\cos(3x) + \frac{1}{3}\int \cos(3x)\,dx \qquad \text{(with } u = x,\ dv = \cos(3x)\,dx\text{)} \\
&= -\frac{1}{3}x\cos(3x) + \frac{1}{9}\sin(3x) + C.
\end{aligned}$$

(c) Here,

$$\begin{aligned}
\int \sin^4 x\,dx &= \int \left(\sin^2 x\right)^2 dx = \int \left(\frac{1}{2}\right)^2 [1-\cos(2x)]^2\,dx = \frac{1}{4}\int \left[1 - 2\cos(2x) + \cos^2(2x)\right] dx \\
&= \frac{1}{4}\int 1\,dx - \frac{1}{2}\int \cos(2x)\,dx + \frac{1}{4}\int \cos^2(2x)\,dx \\
&= \frac{1}{4}x - \frac{1}{4}\sin(2x) + \frac{1}{8}\int [1+\cos(4x)]\,dx = \frac{1}{4}x + \frac{1}{4}\sin(2x) + \frac{1}{8}\left[x + \frac{1}{4}\sin(4x)\right] + C \\
&= \frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C.
\end{aligned}$$

(d) Using the substitution $u = x - 8$, we get

$$\begin{aligned}
\int_8^9 x(x-8)^{1/3}\,dx &= \int_0^1 (u+8)u^{1/3}\,du = \int_0^1 \left(u^{4/3} + 8u^{1/3}\right) du \\
&= \frac{3}{7}u^{7/3} + 6u^{4/3}\Big|_0^1 = \frac{3}{7} + \frac{42}{7} = \frac{45}{7}.
\end{aligned}$$

2. The region is horizontally simple, but not vertically simple. We solve the equations for x in preparation for horizontal slices (i.e, slices at fixed y-values with $0 \le y \le 3$):

left boundary: $x = (y-1)^2 - 1 = y^2 - 2y$
right boundary: $x = y$

Then

$$\text{Area} = \int_0^3 [y - (y^2 - 2y)]\,dy = \int_0^3 (3y - y^2)\,dy = \frac{3}{2}y^2 - \frac{1}{3}y^3\Big|_0^3 = \frac{27}{2} - \frac{27}{3} = \frac{27}{6}.$$

3. (a) Since R is horizontally simple, that provides some motivation for making horizontal slices through R which generate cylindrical shells. The lateral "height" of a cylinder at fixed y is the difference, $(3y - y^2)$ after simplifying, of right and left boundary (see the solution to the previous problem). We obtain

$$V = 2\pi \int_0^3 y(3y - y^2)\, dy = 2\pi \int_0^3 (3y^2 - y^3)\, dy.$$

Perhaps more difficult, but equally valid, is to slice through R vertically and employ the washer method, yielding the sum of x-integrals

$$\begin{aligned} V &= \int_{-1}^0 \pi[(1 + \sqrt{1+x})^2 - (1 - \sqrt{1+x})^2]\, dx + \int_0^3 \pi[(1 + \sqrt{1+x})^2 - x^2]\, dx \\ &= \int_{-1}^0 4\pi \sqrt{1+x}\, dx + \pi \int_0^3 (2 + 2\sqrt{1+x} + x - x^2)\, dx. \end{aligned}$$

(b) Using the method of washers (this time a y-integral), we have

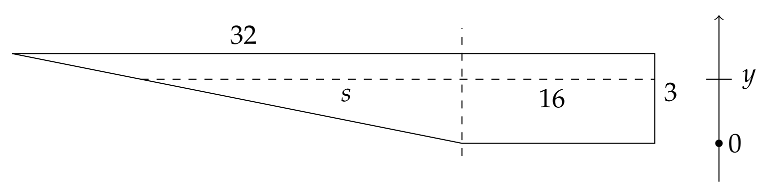
$$V = \pi \int_0^3 \left\{ \left[3 - [(y-1)^2 - 1]\right]^2 - (3-y)^2 \right\} dy = \cdots = \pi \int_0^3 (y^4 - 4y^3 - 3y^2 + 18y)\, dy.$$

If we do this by shells, we get another sum of integrals:

$$\begin{aligned} V &= \int_{-1}^0 2\pi(3-x)[(1 + \sqrt{1+x}) - (1 - \sqrt{1+x})]\, dx + \int_0^3 2\pi(3-x)(1 + \sqrt{1+x} - x)\, dx \\ &= 4\pi \int_{-1}^0 (3-x)\sqrt{1+x}\, dx + 2\pi \int_0^3 (3-x)(1 - x + \sqrt{1+x})\, dx. \end{aligned}$$

4. Viewing the figure, a slice at height y will be a "slab" with length $s + 16$, width 25 and thickness dy. By similar triangles,

$$\frac{s}{32} = \frac{y}{3}, \qquad \text{or} \qquad s = \frac{32}{3} y.$$



Thus, the slab has

$$\begin{aligned} \text{Volume} &= 25\left(\frac{32}{3} y + 16\right) dy \\ \text{Mass} &= 25(1000)\left(\frac{32}{3} y + 16\right) dy \\ \text{Weight} &= 25(9.8)(1000)\left(\frac{32}{3} y + 16\right) dy. \end{aligned}$$

This weight must be lifted from height y to height 3, a distance of $(3 - y)$. So, the desired integral is

$$\text{Work} = \int_0^3 25(9.8)(1000)\left(\frac{32}{3} y + 16\right)(3 - y)\, dy.$$