B Solutions

1. (a) The series is geometric with $r = \sqrt{5}/2 > 1$, and hence divergent.

   (b) The series $\sum_n |\sin n|/n^{3/2}$ has nonnegative terms which are always smaller than the terms of $\sum_n 1/n^{3/2}$. Since the latter is a convergent p-series, the former converges by the direct comparison test. Thus, our original series converges by the absolute convergence test.

2. (a) Making the substitution $x = 2\sec\theta$ (so $dx = 2\sec\theta\tan\theta\,d\theta$), the integral becomes

$$\begin{aligned}
\int \frac{2\sec\theta\tan\theta}{8\sec^3\theta\sqrt{4\sec^2\theta - 4}}\,d\theta &= \frac{1}{8}\int \frac{\sec\theta\tan\theta}{\sec^3\theta\sqrt{\sec^2\theta - 1}}\,d\theta = \frac{1}{8}\int \frac{\sec\theta\tan\theta}{\sec^3\theta\sqrt{\tan^2\theta}}\,d\theta \\
&= \frac{1}{8}\int \frac{\sec\theta\tan\theta}{\sec^3\theta\tan\theta}\,d\theta = \frac{1}{8}\int \cos^2\theta\,d\theta \\
&= \frac{1}{16}\int [1 + \cos(2\theta)]\,d\theta = \frac{1}{16}\left[\theta + \frac{1}{2}\sin(2\theta)\right] + C \\
&= \frac{1}{16}\left[\operatorname{arcsec}\left(\frac{x}{2}\right) + \frac{1}{2}\sin(2\operatorname{arcsec}\left(\frac{x}{2}\right))\right] + C.
\end{aligned}$$

   Note, using the identity $\sin(2\theta) = 2\cos\theta\sin\theta$, one can show that

$$\frac{1}{16}\left[\theta + \frac{1}{2}\sin(2\theta)\right] + C = \frac{1}{16}[\theta + \cos(\theta)\sin(\theta)] + C = \frac{1}{16}\operatorname{arcsec}\left(\frac{x}{2}\right) + \frac{\sqrt{x^2-4}}{8x^2} + C,$$

   but the above answer is just fine.

   (b) Since

$$\frac{17-3x}{x^2-2x-3} = \frac{17-3x}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1},$$

   we have

$$17 - 3x = A(x+1) + B(x-3).$$

   This equation must hold for all x—in particular, at

$$x = -1: \quad \text{giving} \quad 17 - 3(-1) = A(0) + B(-1-3) \quad \Rightarrow \quad B = -5,$$

   and at

$$x = 3: \quad \text{giving} \quad 17 - 3(3) = A(3+1) + B(0) \quad \Rightarrow \quad A = 2.$$

   Thus, our integral equals

$$\int \frac{2}{x-3}\,dx - \int \frac{5}{x+1}\,dx = 2\ln|x-3| - 5\ln|x+1| + C.$$

3. We have

$$\sum_{n=0}^{\infty} \frac{3^n}{2^{2n+1}} = \frac{1}{2} + \frac{3}{8} + \frac{9}{32} + \cdots = \frac{1}{2}\left[1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \cdots\right],$$

   a geometric series with $r = 3/4$ (thus convergent) whose sum is

$$\frac{1}{2}\cdot\frac{1}{1-3/4} = 2.$$

4. Since $f'(x) = 3x^{1/2}$, we have arc length given by the integral

$$\int_0^7 \sqrt{1+9x}\,dx = \frac{1}{9}\int_1^{64} u^{1/2}\,du = \frac{2}{27}\,u^{3/2}\Big|_1^{64} = \frac{2}{27}(8^3 - 1) = \frac{1022}{27} \doteq 37.85.$$

5. We have derivatives

$$
\begin{aligned}
f^{(0)}(x) &= x^{2/3} & \Rightarrow && f^{(0)}(1) &= 1. \\
f'(x) &= \frac{2}{3}x^{-1/3} & \Rightarrow && f'(1) &= \frac{2}{3}. \\
f''(x) &= -\frac{2}{9}x^{-4/3} & \Rightarrow && f''(1) &= -\frac{2}{9}.
\end{aligned}
$$

Thus, the desired Taylor polynomial is

$$T_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 = 1 + \frac{2}{3}(x-1) - \frac{1}{9}(x-1)^2.$$

6. If we first find an antiderivative using the substitution $u = \ln x$ (so $du = dx/x$)

$$\int \frac{dx}{x(\ln x)^{5/4}} = \int \frac{du}{u^{5/4}} = \int u^{-5/4}\,du = -4u^{-1/4} + C = -\frac{4}{(\ln x)^{1/4}} + C,$$

then we have

$$\int_2^\infty \frac{dx}{x(\ln x)^{5/4}} = \lim_{A\to\infty} \int_2^A \frac{dx}{x(\ln x)^{5/4}} = \lim_{A\to\infty} \left[-\frac{4}{(\ln x)^{1/4}}\right]_2^A = \lim_{A\to\infty} \left[\frac{4}{(\ln 2)^{1/4}} - \frac{4}{(\ln A)^{1/4}}\right] = \frac{4}{(\ln 2)^{1/4}}.$$

Thus, the integral converges (to the value specified).