A Solutions

1. (a) Employing the ratio test, we have

$$\frac{|x-2|^{n+1}/(3^{2(n+1)}(n+1))}{|x-2|^n/(3^{2n}n)} = \frac{|x-2|^{n+1}}{|x-2|^n}\cdot\frac{3^{2n}}{3^{2n+2}}\cdot\frac{n}{n+1} = \frac{|x-2|}{3^2}\cdot\frac{n}{n+1} \to \frac{|x-2|}{9} \quad \text{as } n\to\infty.$$

The ratio test guarantees convergence when this (positive) fraction is less than 1, and divergence when it is greater than 1. Thus, we have convergence when

$$\frac{|x-2|}{9} < 1 \qquad \Rightarrow \qquad |x-2| < 9.$$

So, the series is centered at 2, and has radius of convergence 9.

(b) The endpoints of the interval of convergence—the numbers which are exactly 9 units away from 2—are (-7) and 11. We substitute these in for x and determine if the resulting series converge.
At $x=-7$:

$$\sum_{n=1}^{\infty}\frac{(-1)^n(-9)^n}{(3^2)^n n} = \sum_{n=1}^{\infty}\frac{[(-1)^2]^n 9^n}{9^n n} = \sum_{n=1}^{\infty}\frac{1}{n},$$

giving us the harmonic series (p-series with $p=1$), which is divergent. So $x=-7$ is not included in the IOC.
At $x=11$:

$$\sum_{n=1}^{\infty}\frac{(-1)^n 9^n}{(3^2)^n n} = \sum_{n=1}^{\infty}\frac{(-1)}{n},$$

giving us the alternating harmonic series, which converges by the Alternating Series Test.
Thus, the IOC is $(-7, 11]$.

2. (a) The coordinates have derivatives

$$x'(t) = (2t+3)^{1/2} \qquad \text{and} \qquad y'(t) = 1+t.$$

So,

$$\frac{dy}{dx} = \frac{1+t}{\sqrt{2t+3}} \qquad \Rightarrow \qquad \left.\frac{dy}{dx}\right|_{t=3} = \frac{4}{3},$$

giving the slope of the tangent line. The point of tangency is $(x(3), y(3)) = \left(\frac{9^{3/2}}{3}, 3+\frac{9}{2}\right) = \left(9, \frac{15}{2}\right)$.
Employing the point-slope formula for lines, the equation is

$$y-\frac{15}{2} = \frac{4}{3}(x-9), \qquad \text{or} \qquad y = \frac{4}{3}x-\frac{9}{2}.$$

(b) The length of the arc is

$$\int_0^3\sqrt{[x'(t)]^2+[y'(t)]^2}\,dt = \int_0^3\sqrt{2t+3+(1+t)^2}\,dt = \int_0^3\sqrt{t^2+4t+4}\,dt = \int_0^3\sqrt{(t+2)^2}\,dt$$

$$= \int_0^3 (t+2)\,dt = \left.\frac{1}{2}t^2+2t\right|_0^3 = \frac{9}{2}+6 = \frac{21}{2}.$$

3. (a) *(4 pts)* We are facing 45° into Quadrant I but heading backwards 3 units from the origin with that bearing. The rectangular coordinates of the point we reach are $(-3/\sqrt{2}, -3/\sqrt{2})$.

(b) *(4 pts)* The line segment from the origin to the given point is the hypotenuse of a right triangle whose obtuse angle (back to the positive x-axis) is $5\pi/6$. So, the reference angle is $\pi/6$ (or $30°$), and when the shorter leg is of length 1 in a $30° - 60° - 90°$ triangle, the length of the hypotenuse is 2. So, one set of polar coordinates (of course, there are infinitely many others) identifying this point is $(2, 5\pi/6)$.

(c) *(6 pts)* Using that $x = r\cos\theta$ and $y = r\sin\theta$, we write

$$
\begin{aligned}
x^2 + (y-2)^2 = 4 \quad &\Rightarrow \quad (r\cos\theta)^2 + (r\sin\theta - 2)^2 = 4 \\
&\Rightarrow \quad r^2\cos^2\theta + r^2\sin^2\theta - 4r\sin\theta + 4 = 4 \\
&\Rightarrow \quad r^2(\cos^2\theta + \sin^2\theta) - 4r\sin\theta = 0 \\
&\Rightarrow \quad r^2 - 4r\sin\theta = 0 \\
&\Rightarrow \quad r(r - 4\sin\theta) = 0 \\
&\Rightarrow \quad r = 4\sin\theta.
\end{aligned}
$$

4. (a) *(8 pts)* We have

$$f(x) = 3 \cdot \frac{1}{1-(-4x)} = 3 \cdot \sum_{n=0}^{\infty}(-4x)^n = \sum_{n=0}^{\infty}(-1)^n(3)(4^n)x^n = 3 - 12x + 48x^2 - 192x^3 + \cdots .$$

Convergence happens when

$$|-4x| < 1 \qquad \Rightarrow \qquad |x| < \frac{1}{4}.$$

So, the radius of convergence is $R = 1/4$.

(b) *(6 pts)* Applying term-by-term differentiation to the series in part (a), we get

$$-\frac{12}{(4x+1)^2} = \frac{d}{dx}\left(3 - 12x + 48x^2 - 192x^3 + \cdots\right) = -12 + 96x - 576x^2 + \cdots = \sum_{n=1}^{\infty}(-1)^n(3n)4^n x^{n-1}.$$

5. (a) $T_3(x)$ includes the terms with powers of x up to and including 3. So,

$$T_3(x) = 1 - \frac{1}{2!}x^2.$$

(b) We have

$$-1 + \cos\sqrt{x} = -1 + 1 - \frac{(\sqrt{x})^2}{2!} + \frac{(\sqrt{x})^4}{4!} - \frac{(\sqrt{x})^6}{6!} + \cdots = -\frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \cdots = \sum_{n=1}^{\infty}(-1)^n\frac{x^n}{(2n)!}.$$

(c) Dividing the series of part (b) by x and taking the limit, we have

$$\lim_{x\to 0}\frac{1}{x}\left(-\frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \cdots\right) = \lim_{x\to 0}\left(-\frac{1}{2!} + \frac{x}{4!} - \frac{x^2}{6!} + \cdots\right) = -\frac{1}{2} + 0 + 0 + \cdots = -\frac{1}{2}.$$