B Solutions

1. (a) Employing the ratio test, we have

$$\frac{|x-1|^{n+1}/(2^{2(n+1)}(n+1))}{|x-1|^n/(2^{2n}n)} = \frac{|x-1|^{n+1}}{|x-1|^n}\cdot\frac{2^{2n}}{2^{2n+2}}\cdot\frac{n}{n+1} = \frac{|x-1|}{2^2}\cdot\frac{n}{n+1} \to \frac{|x-1|}{4} \quad \text{as } n\to\infty.$$

The ratio test guarantees convergence when this (positive) fraction is less than 1, and divergence when it is greater than 1. Thus, we have convergence when

$$\frac{|x-1|}{4} < 1 \qquad \Rightarrow \qquad |x-1| < 4.$$

So, the series is centered at 1, and has radius of convergence 4.

(b) The endpoints of the interval of convergence—the numbers which are exactly 4 units away from 1—are (-3) and 5. We substitute these in for x and determine if the resulting series converge. At $x=-3$:

$$\sum_{n=1}^{\infty}\frac{(-4)^n}{(2^2)^n n} = \sum_{n=1}^{\infty}\frac{(-1)^n 4^n}{4^n n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n},$$

giving us the alternating harmonic series, which converges by the Alternating Series Test. Thus $x=-3$ is in the IOC.

At $x=5$:

$$\sum_{n=1}^{\infty}\frac{4^n}{(2^2)^n n} = \sum_{n=1}^{\infty}\frac{1}{n},$$

giving us the harmonic series (p-series with $p=1$), which is divergent. So $x=5$ is not included in the IOC.

Thus, the IOC is $[-3,5)$.

2. (a) The coordinates have derivatives

$$x'(t) = 2-3t \qquad \text{and} \qquad y'(t) = (6t-3)^{1/2}.$$

So,

$$\frac{dy}{dx} = \frac{\sqrt{6t-3}}{2-3t} \qquad \Rightarrow \qquad \left.\frac{dy}{dx}\right|_{t=2} = -\frac{3}{4},$$

giving the slope of the tangent line. The point of tangency is $(x(3),y(3)) = \left(4-6,\frac{9^{3/2}}{9}\right) = (-2,3)$. Employing the point-slope formula for lines, the equation is

$$y-3 = -\frac{3}{4}(x+2), \qquad \text{or} \qquad y = -\frac{3}{4}x+\frac{3}{2}.$$

(b) The length of the arc is

$$\begin{aligned}\int_0^4\sqrt{[x'(t)]^2+[y'(t)]^2}\,dt &= \int_0^4\sqrt{(2-3t)^2+6t-3}\,dt = \int_0^4\sqrt{9t^2-6t+1}\,dt = \int_0^4\sqrt{(3t-1)^2}\,dt\\ &= \int_0^4|3t-1|\,dt = \int_0^{1/3}(1-3t)\,dt+\int_{1/3}^4(3t-1)\,dt\\ &= \left[t-\frac{3}{2}t^2\right]_0^{1/3}+\left[\frac{3}{2}t^2-t\right]_{1/3}^4 = \left(\frac{1}{3}-\frac{1}{6}\right)+\left[(24-4)-\left(\frac{1}{6}-\frac{1}{3}\right)\right]\\ &= 20+\frac{1}{3} = \frac{61}{3}.\end{aligned}$$

3. (a) *(4 pts)* We are facing 135° relative to the positive x-axis, into Quadrant II, but heading backwards 2 units from the origin with that bearing. The rectangular coordinates of the point we reach are $(2/\sqrt{2}, -2/\sqrt{2})$, or $(\sqrt{2}, -\sqrt{2})$.

   (b) *(4 pts)* The line segment from the origin to the given point is the hypotenuse of a right triangle whose obtuse angle (back to the positive x-axis) is $4\pi/3$. So, the reference angle is $\pi/3$ (or 60°), and when the shorter leg is of length 1 in a $30° - 60° - 90°$ triangle, the length of the hypotenuse is 2. So, one set of polar coordinates (of course, there are infinitely many others) identifying this point is $(2, 4\pi/3)$.

   (c) *(6 pts)* Using that $x = r\cos\theta$ and $y = r\sin\theta$, we write

$$\begin{aligned}
(x-3)^2 + y^2 = 9 \qquad &\Rightarrow \qquad (r\cos\theta - 3)^2 + (r\sin\theta)^2 = 9\\
&\Rightarrow \qquad r^2\cos^2\theta - 6r\cos\theta + 9 + r^2\sin^2\theta = 9\\
&\Rightarrow \qquad r^2(\cos^2\theta + \sin^2\theta) - 6r\cos\theta = 0\\
&\Rightarrow \qquad r^2 - 6r\cos\theta = 0\\
&\Rightarrow \qquad r(r - 6\cos\theta) = 0\\
&\Rightarrow \qquad r = 6\cos\theta.
\end{aligned}$$

4. (a) *(8 pts)* We have

$$f(x) \;=\; 5\cdot\frac{1}{1-(-3x)} \;=\; 5\cdot\sum_{n=0}^{\infty}(-3x)^n \;=\; \sum_{n=0}^{\infty}(-1)^n(5)(3^n)x^n \;=\; 5 - 15x + 45x^2 - 135x^3 + \cdots.$$

   Convergence happens when

$$|-3x| < 1 \qquad \Rightarrow \qquad |x| < \frac{1}{3}.$$

   So, the radius of convergence is $R = 1/3$.

   (b) *(6 pts)* Applying term-by-term differentiation to the series in part (a), we get

$$-\frac{12}{(4x+1)^2} \;=\; \frac{d}{dx}\left(5 - 15x + 45x^2 - 135x^3 + \cdots\right) \;=\; -15 + 90x - 405x^2 + \cdots \;=\; \sum_{n=1}^{\infty}(-1)^n(5n)3^n x^{n-1}.$$

5. (a) $T_6(x)$ includes the terms with powers of x up to and including 6. So,

$$T_6(x) \;=\; x - \frac{1}{3!}x^3 - \frac{1}{5!}x^5.$$

   (b) We have

$$-x^2 + \sin(x^2) \;=\; -x^2 + x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots \;=\; -\frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots \;=\; \sum_{n=1}^{\infty}(-1)^n\frac{x^{4n+2}}{(2n+1)!}.$$

   (c) Dividing the series of part (b) by x^6 and taking the limit, we have

$$\lim_{x\to 0}\frac{1}{x^6}\left(-\frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots\right) \;=\; \lim_{x\to 0}\left(-\frac{1}{3!} + \frac{x^4}{5!} - \frac{x^8}{7!} + \cdots\right) \;=\; -\frac{1}{6} + 0 + 0 + \cdots \;=\; -\frac{1}{6}.$$