MATH 231: Differential Equations with Linear Algebra

Hand-Checked Assignment #4, due date: Tues., Mar. 30, 2021

Write up, carefully and legibly, your solutions to the following problems. While you do not need to present one problem per page, please do not put problems side-by-side (i.e., no two-column format). To submit your work it must be

- scanned (all pages) to a single .pdf file (one multi-page file containing all graded problems).
- submitted to https://www.gradescope.com as **hc04**.

$\underline{\star 31}$ The following facts hold about the given matrices and the corresponding homogeneous linear 1st order systems of DEs $\mathbf{x}' = \mathbf{A}\mathbf{x}$:

1. When $\mathbf{A} = \begin{bmatrix} 5 & 2 & -12 \\ -4 & -1 & 12 \\ 0 & -2 & -7 \end{bmatrix}$:

λ	basis eigenvectors	solns for fund'l set
-3 (AM=GM=1)	$\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$	$e^{-3t}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$
-1 (AM=GM=1)	$\begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}$	$e^{-t}\begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}$
1 (AM=GM=1)	$\begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}$	$e^{t}\begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}$

General solution:

$$\mathbf{x}(t) = c_1 e^{-3t}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix} + c_3 e^{t}\begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 2e^{-3t} & 3e^{-t} & 5e^{t} \\ -2e^{-3t} & -3e^{-t} & -4e^{t} \\ e^{-3t} & e^{-t} & e^{t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

2. When $\mathbf{A} = \begin{bmatrix} 5.5 & -1.5 & 1.5 \\ 0.5 & -0.5 & -1.5 \\ -1 & -1 & 2 \end{bmatrix}$:

λ	basis eigenvectors	solns for fund'l set
4 (AM=2, GM=1)	$\begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix}$	$e^{4t}\begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix}$, $e^{4t}\left(t\begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}\right)$
−1 (AM=GM=1)	$\begin{bmatrix} 3 \\ 21 \\ 8 \end{bmatrix}$	$e^{-t}\begin{bmatrix} 3 \\ 21 \\ 8 \end{bmatrix}$

General solution:

$$\mathbf{x}(t) \;=\; c_1 e^{-t}\begin{bmatrix} 3 \\ 21 \\ 8 \end{bmatrix} + c_2 e^{4t}\begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix} + c_3 e^{4t}\begin{bmatrix} -3t-2 \\ -t \\ 2t \end{bmatrix}.$$

3. When $\mathbf{A} = \begin{bmatrix} -4 & -4 & -7 \\ 7 & 10 & 18 \\ -3 & -5 & -9 \end{bmatrix}$:

λ	basis eigenvectors	solns for fund'l set
−1 (AM=3, GM=1)	$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$	$e^{-t}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $e^{-t}\left(t\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}\right)$, $e^{-t}\left(\frac{1}{2}t^2\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + t\begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix}\right)$

General solution:

$$\mathbf{x}(t) \;=\; c_1 e^{-2t}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} t-3 \\ t+2 \\ -t \end{bmatrix} + c_3 e^{-t}\begin{bmatrix} (1/2)t^2 - 3t + 5 \\ (1/2)t^2 + 2t - 3 \\ (1/2)t^2 \end{bmatrix}.$$

This problem is investigatory in nature, delving into questions which have not all been addressed in class.

(a) For the matrices given in numbers 2 and 3, show that each of the vector functions included in a fundamental set (3 for each of the two matrices **A**) are, indeed, solutions of the homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

(b) Consider the matrix **A** in number 1, along with its eigenpair $\lambda = -3$, $\mathbf{v} = \langle 2, -2, 1\rangle$. Is there a vector $\mathbf{u}$ which solves $(\mathbf{A} + 3\mathbf{I})\mathbf{u} = \mathbf{v}$? What do you conclude about the existence of a solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ of the form

$$(t\mathbf{v} + \mathbf{u})\, e^{-3t}, \qquad \text{with} \qquad (\mathbf{A} + 3\mathbf{I})\mathbf{v} = \mathbf{0} \;\text{ and }\; (\mathbf{A} + 3\mathbf{I})\mathbf{u} = \mathbf{v}?$$

(c) Consider the matrix $\mathbf{A}$ in number 2, along with its degenerate eigenvalue $\lambda = 4$. We have that $\mathbf{v} = \langle -3, -1, 2\rangle$, is a basis for the corresponding 1-dimensional (since GM=1) eigenspace, and that $\mathbf{u} = \langle -2, 0, 0\rangle$ satisfies $(\mathbf{A} - 4\mathbf{I})\mathbf{u} = \mathbf{v}$. Is there a vector $\mathbf{w}$ which solves $(\mathbf{A} - 4\mathbf{I})\mathbf{w} = \mathbf{u}$? What do you conclude about the existence of a solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ of the form

$$\left(\frac{1}{2!}t^2\mathbf{v} + t\mathbf{u} + \mathbf{w}\right)e^{4t}, \qquad \text{with} \qquad (\mathbf{A}-4\mathbf{I})\mathbf{v} = \mathbf{0},\ \ (\mathbf{A}-4\mathbf{I})\mathbf{u} = \mathbf{v}\ \text{ and }\ (\mathbf{A}-4\mathbf{I})\mathbf{w} = \mathbf{u}?$$

(d) Under what conditions will you look for a solution to a homogeneous $\mathbf{x}' = \mathbf{A}\mathbf{x}$ of the form

$$(t\mathbf{v} + \mathbf{u})\, e^{\lambda t}?$$

Under what conditions will you look for a solution of the form

$$\left(\frac{1}{2!}t^2\mathbf{v} + t\mathbf{u} + \mathbf{w}\right)e^{\lambda t} \qquad \text{or} \qquad \left(\frac{1}{3!}t^3\mathbf{v} + \frac{1}{2!}t^2\mathbf{u} + t\mathbf{w} + \mathbf{z}\right)e^{\lambda t}?$$

★32 On pp. 188-189 of ODELA we learn about a 1st order linear DE system model for quantities of lead stored in 1) the blood, 2) body tissues, and 3) bones. The model is

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \mathbf{A}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} I_L(t) \\ 0 \\ 0 \end{bmatrix}, \qquad \text{where} \qquad \mathbf{A} = \begin{bmatrix} -13/360 & 272/21875 & 7/200000 \\ 1/90 & -1/35 & 0 \\ 7/1800 & 0 & -7/200000 \end{bmatrix}.$$

(a) Approximate eigenpairs for the matrix $\mathbf{A}$ are provided at the top of p. 190. Use them to write the homogeneous solution—i.e., the solution in the case the influx into the bloodstream of lead from the environment $I_L(t) = 0$.

(b) If $I_L(t) = 0$ for a person previously poisoned with lead, what aspect of the model or solution indicates that the lead will be flushed out over time?

(c) Continue assuming that $I_L(t) = 0$, but suppose we have initial conditions $x_1(0) = 50$, $x_2(0) = 0$ and $x_3(0) = 0$; that is, we start with 50 units of lead in the blood and none in tissue nor bone. Solve the (homogeneous) IVP, and use it to write a formula for the amount $x_3(t)$ of lead in the bones. Find the approximate time t (in days) at which the level of lead in the bones is at its peak value. [Give your answer accurate to the tenths place.] Also, find the approximate time, following that peak, when the lead level in the bones has receded to no more than 0.5 units.

★33 From ODELA Section 3.6, pp. 181–183, do Exercise 3.6.5.

★34 From ODELA Section 3.6, pp. 181–183, do Exercise 3.6.8.