MATH 231: Differential Equations with Linear Algebra

Hand-Checked Assignment #5, due date: Thurs., Apr. 22, 2021

Write up, carefully and legibly, your solutions to the following problems. While you do not need to present one problem per page, please do not put problems side-by-side (i.e., no two-column format). To submit your work it must be

- scanned (all pages) to a single .pdf file (one multi-page file containing all graded problems).
- submitted to `https://www.gradescope.com` as **hc05**.

★35 For $\omega \neq \omega_0$, consider the undamped, forced spring differential equation with *resting* initial conditions:

$$mu'' + ku = F_0 \cos(\omega t), \qquad u(0) = 0, \; u'(0) = 0.$$

(a) Show that the solution of this IVP is

$$u(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)}\Big[\cos(\omega t) - \cos(\omega_0 t)\Big].$$

(b) Solve the system of two equations in the two unknowns A, B

$$A + B = \omega_0 t, \qquad A - B = \omega t.$$

Then use the trigonometric identities

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta, \qquad \cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

to rewrite the factor $\cos(\omega t) - \cos(\omega_0 t)$ as $2\sin A \sin B$.

(c) Take $m = 1$, $F_0 = 1$, $\omega_0 = 441$ and $\omega = 439$ in the solution to part (a). Graph the functions

$$y = \sin(440t) \qquad \text{and} \qquad y = \sin t,$$

and use them to explain the look of the graph of the solution to part (a).

★36 Consider the undamped, forced spring differential equation

$$mu'' + ku = F_0 \cos(\omega_0 t),$$

where the periodic forcing function has frequency matching $\omega_0 = \sqrt{k/m}$.

(a) Explain why, after the method of Undetermined Coefficients, you would propose the form

$$u_p(t) \;=\; At\cos(\omega_0 t) + Bt\sin(\omega_0 t)$$

for a particular solution rather than

$$u_p(t) \;=\; A\cos(\omega_0 t) + B\sin(\omega_0 t).$$

(b) Use the proposed u_p from part (a) to find the general solution $u_h(t) + u_p(t)$ of the differential equation. As a spoiler, you're shooting for

$$u(t) \;=\; c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{F_0}{2m\omega_0}\,t\sin(\omega_0 t).$$

(c) In class we investigated the undamped, forced spring vibrations under the assumption of a periodic forcing function whose frequency ω is *different* than ω_0. In that case,

$$u_p(t) \;=\; \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos(\omega t).$$

Explain how the qualitative differences in the forms of $u_p(t)$ imply that an **undamped** mass-spring system might not break when forced at frequency $\omega \neq \omega_0$, but assuredly will break if $\omega = \omega_0$.

★37 [Part (f) below is optional.] Consider the 2nd order linear nonhomogeneous DE

$$y'' + 2by' + \omega_0^2 y \;=\; f_0\cos(\omega t), \tag{1}$$

where b, ω_0, and f_0 are all positive constants. We have seen that the DE (1) can be a model for a damped mass-spring system or an RLC circuit. In this problem, we investigate the influence of the forcing frequency on amplitude and phase shift.

(a) Guessing (i.e., using the Method of Undetermined Coefficients) the form $y_p(t) = A\cos(\omega t) + B\sin(\omega t)$ for the steady-state solution, show that

$$A \;=\; \frac{(\omega_0^2 - \omega^2)f_0}{\Delta} \qquad \text{and} \qquad B \;=\; \frac{2b\omega f_0}{\Delta},$$

where $\Delta = (\omega_0^2 - \omega^2)^2 + 4b^2\omega^2$.

(b) Determine formulas for R and ϕ so that $y_p(t) = R\cos(\omega t - \phi)$.

(c) Take $f_0 = 1$, $b = 1$, and $\omega_0 = 15$. Plot R and ϕ as functions of the forcing frequency ω.

(d) Repeat the previous part, but now with $f_0 = 1$, $b = 8$, and $\omega_0 = 20$.

(e) Using the formula for R you found in part (b) (no specific falues for f_0, b nor ω_0) and calculus, find the value of ω which maximizes amplitude R.

(f) What information do the phase functions $\phi(\omega)$ plotted in parts (c) and (d) tell? Give a physical interpretation that presumes the transient part of the solution has died out.

★38 Consider the 2nd-order system

$$\begin{aligned} \frac{d^2u_1}{dt^2} &= \frac{k_2}{m_1}(u_2 - u_1) + \frac{c_2}{m_1}\left(\frac{du_2}{dt} - \frac{du_1}{dt}\right) - \frac{k_1}{m_1}u_1 - \frac{c_1}{m_1}\frac{du_1}{dt}, \\ \frac{d^2u_2}{dt^2} &= \frac{f(t)}{m_2} - \frac{k_2}{m_2}(u_2 - u_1) - \frac{c_2}{m_2}\left(\frac{du_2}{dt} - \frac{du_1}{dt}\right). \end{aligned}$$

which may (it has been proposed by someone on the internet, but I have not verified it) accurately model the mass-spring assembly with dashpots (shock absorbers) pictured at the website

http://upload.wikimedia.org/wikipedia/commons/f/fd/Mass-Spring-Damper_%282_body_system%29.svg

It seems the displacements from equilibrium positions of masses m_1, m_2 are denoted by u_1, u_2 respectively. The spring constants are k_1, k_2, and c_1, c_2 are the damping constants associated with the dashpots. There appears, as well, to be an external force $f(t)$ applied to the mass on the right.

Convert this model (whether it models what it purports to or not) to a first order linear system of DEs in the form $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}(t)$, specifying the entries of $\mathbf{A}$ and $\mathbf{b}(t)$.

★39 From ODELA Section 4.2, pp. 215–217, do Exercise 4.2.6.