Matrix, $m \times n$ (m = # of rows, n = # of cols)

table of numbers (real)

vector in $\mathbb{R}^m$ is $m \times 1$ matrix

Define: matrix - vector product

$$\underset{2\times 4}{\begin{bmatrix} 3 & 1 & 2 & 4 \\ 1 & -1 & 0 & 5 \end{bmatrix}} \cdot \underset{\in \mathbb{R}^4}{\begin{bmatrix} 1 \\ -1 \\ 2 \\ 4 \end{bmatrix}} := \overbrace{1\begin{bmatrix} 3 \\ 1 \end{bmatrix} + (-1)\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 4 \\ 5 \end{bmatrix}}^{\text{linear comb. of cols. in matrix}}$$

$$= \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 16 \\ 20 \end{bmatrix}$$

$$= \begin{bmatrix} 22 \\ 22 \end{bmatrix}$$

$$A_{m\times n} \underset{\in \mathbb{R}^n}{\vec{v}} := \text{linear comb. of cols. of } A \text{ using weights from } \vec{v}.$$

Problem from yesterday

1A Is $\begin{bmatrix} -2 \\ 4 \\ 15 \end{bmatrix}$ in span of $\left\{ \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 9 \end{bmatrix} \right\}$

can be seen in other "lights"

$$\left\{ \begin{aligned} 3x_1 - x_2 + x_3 &= -2 \\ -x_1 + x_2 + x_3 &= 4 \\ 5x_1 + 2x_2 + 9x_3 &= 15 \end{aligned} \right\}$$

1B Does this system of equations have a solution?

1C Do weights x_1, x_2, x_3 exist so that

$$x_1\begin{bmatrix}3\\-1\\5\end{bmatrix} + x_2\begin{bmatrix}-1\\1\\2\end{bmatrix} + x_3\begin{bmatrix}1\\1\\9\end{bmatrix} = \begin{bmatrix}-2\\4\\15\end{bmatrix}$$

1D

$$\begin{bmatrix}3 & -1 & 1\\-1 & 1 & 1\\5 & 2 & 9\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}-2\\4\\15\end{bmatrix}$$

Note: 1A - 1D all the same problem

Method to solve: GE

$$\left[\begin{array}{ccc|c}3 & -1 & 1 & -2\\-1 & 1 & 1 & 4\\5 & 2 & 9 & 15\end{array}\right] \underset{\sim}{\tfrac{1}{3}r_1 \to r_1} \left[\begin{array}{c|c}\text{omitted} & \end{array}\right]$$

$$\begin{array}{c} r_1 + r_2 \to r_2 \\ \sim \\ -5r_1 + r_3 \to r_3 \end{array} \left[\begin{array}{ccc|c}1 & -1/3 & 1/3 & -2/3\\0 & 2/3 & 4/3 & 10/3\\0 & 11/3 & 22/3 & 55/3\end{array}\right]$$

$$\underset{\sim}{\tfrac{3}{2}r_2 \to r_2} \begin{bmatrix}\text{omit}\end{bmatrix} \quad \underset{\sim}{-\tfrac{11}{3}r_2 + r_3 \to r_3} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & \text{free var.} \end{array} \\ \left[\begin{array}{ccc|c}1 & -1/3 & 1/3 & -2/3\\0 & 1 & 2 & 5\\0 & 0 & 0 & 0\end{array}\right] \end{array}$$

echelon form

<u>echelon form:</u> In each row, first nonzero entry (called a pivot) appears further to the right than in preceding rows.

Counterpart:

$$\left.\begin{aligned} x_1 - \tfrac{1}{3}x_2 + \tfrac{1}{3}x_3 &= -\tfrac{2}{3} \\ x_2 + 2x_3 &= 5 \end{aligned}\right\} \text{ 2 meaningful constraints}$$

$$\boxed{0 = 0}$$ (not too helpful, but true)

variables corresp. to freedoms
eqns. " " constraints

Take x_3 as free (any $\mathbb{R}$)

Solve for x_2, x_1 (pivot vars.) in terms of the free var. x_3

$$x_2 = 5 - 2x_3$$

$$x_1 = \frac{1}{3}x_2 - \frac{1}{3}x_3 - \frac{2}{3}$$

$$= \frac{1}{3}(5 - 2x_3) - \frac{1}{3}x_3 - \frac{2}{3}$$

$$= \frac{5}{3} - \frac{2}{3}x_3 - \frac{1}{3}x_3 - \frac{2}{3} = 1 - x_3$$

Goal was to find weights x_1, x_2, x_3 (arranged in vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$)

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 - x_3 \\ 5 - 2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

where x_3 is any $\mathbb{R}$.

Expect:

- You can describe sequence of EROs leading to echelon form
- In practice, you'll employ a "calculator" button RREF

RREF = reduced row echelon form

= echelon form + additional requirements

all pivots = 1

each pivot col. has just one nonzero entry

RREF for our augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 + x_3 = 1$

$x_2 + 2x_3 = 5$

x_3 free (any $\mathbb{R}$)

$x_2 = 5 - 2x_3$ like earlier

$x_1 = 1 - x_3$ like before, but easier

A possible result in RREF is the following: augmented column is pivot column

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} & 1 \end{array}\right] \longrightarrow 0 = 1$$

4 unknowns

3 eqns. at start, boil down to one that cannot be met!

$0 = 1$ nonsense! <u>No solution</u>

Some conclusions:

- Can figure out which are pivot columns from either (row) echelon form or RREF
- If there is an augmented column which is a pivot column, then the problem is **inconsistent** (has no solutions)
- If there is an augmented column and it is not a pivot column, the problem is **consistent** (has at least one solution)
- If there is a free column among those to the left of the augmentation bar and the problem is consistent, then there are infinitely many solutions.
- If there are no free columns among those to the left of the augmentation bar and the problem is consistent, then there is exactly one solution (at least one choice of weights). This situation can arise only in the case where the number of variables matches the number of equations---that is, it is necessary for the number of variables n to match the number of equations m. But $m=n$ is not a sufficient condition for there to be exactly one solution, as it is possible for one or more of the variables to be free (as occurred in today's class example).