Math 231, Wed 10-Feb-2021 -- Wed 10-Feb-2021
Differential Equations and Linear Algebra
Spring 2020

Wednesday, February 10th 2021

Topic:: Matrix inverses
Topic:: Column space
HW::   HC01 due Feb. 15

Last time
 - we defined matrix multiplication
 - saw that matrix multiplication is not commutative
    may make sense as AB, but not as BA
    for both A_{mxn} B_{qxp} and BA to make sense, need n=q and m=p
      even then AB is mxm while BA is nxn, so equality is possible only if m=n
      ==> A, B square, same dimension is necessary, but not sufficent for AB=BA

Building on this
 - the identity matrices
    square
    acts role of multiplicative identity
 - the problem Ax = b
    ax = b from earliest algebra class provides inspiration x = a^{-1) b
    define inverse of square matrix A:
      another square matrix C
      AC = CA = I
    solution of Ax = b, when A^{-1} exists, is x = A^{-1} b
 - method for finding A^{-1}
    argue that [A | I] row-reduced turns to [I | A^{-1}]
    presumes (and only works if)
      A is square
      every column of A is a pivot column
    If goal is to solve Ax = b, is finding A^{-1} as intermediate step worth it?
    Do by hand for 2-by-2 A = [a b; c d]

One new term:
 - so far

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    Is b in the span of set of vectors
    Are there weights such that a linear combination of vectors produces b?
    Does a system of m equations in n unknowns have a solution?
    Is Ax = b consistent?
 - new
    Is b in the column space of A
 - Determine whether any/all are true by doing GE on augmented  [A | b]

Can use GE to describe the column space of A_{mxn}
 - if RREF(A) has a pivot in every row, then col(A) = R^m
 - when RREF(A) has a row of zeros at bottom, the story is more interesting
    example:  A = [2 -1 5; 1 1 1; -1 2 -4]

Notes:
 - 0 as a vector
 - A(u+v) = Au + Av,   and   A(cu) = c(Au)
 - meaning of "A is nonsingular (invertible)"
    Since AB = BA = I,  if B has columns B_i, then
    Cols:  AB_i = e_i,  so solve for B_i via GE on [A | e_i]
      ==>  get all cols of B simultaneously by applying GE to  [A | I]
    Inverse exists  <==>  A is square with no free cols
                    <==>  rref(A) = I
 - more examples: inverse (when it exists) of
    A = [2 2 1 2; 3 1 2 1; 1 -1 2 1]   (cannot, since non-square)
    A = [2 2 1; 3 1 2; 1 -1 2]     (answer: [-1 5/4 -3/4; 1 -3/4 1/4; 1 -1 1])
    A = [2 2 1; 3 1 2; 1 -1 1]     (answer: singular matrix)
   Use middle result above to solve the system of equations
     2x + 2y +  z =  3
     3x +  y + 2z = -8
      x -  y + 2z =  4
   Because last matrix A was singular, funny behavior can happen:
     That A*[2; -1; 1] = [3; 7; 4] = A*[5;-2;-3]  (no left-cancellation)
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Last time: $\underset{m \times n}{A} \cdot \underset{q \times p}{B}$

"AB makes sense" requires $q = n$ $\Rightarrow$ AB is $m \times p$ matrix

"BA makes sense" " $p = m$ $\Rightarrow$ BA is $q \times n$ matrix

$\Rightarrow$ If both make sense $(q = n,\ p = m)$ then

$$\left.\begin{array}{l} AB \text{ is } m \times m \\ BA \text{ is } n \times n \end{array}\right\} \text{ square matrices}$$

For $AB = BA$ further require $m = n$ (necessary, but not sufficient condition for $AB = BA$)

Identity matrices I_n

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \ldots \quad I_n \text{ is } n \times n \text{ same form}$$

Whenever AI or IA make sense, the product is A.

$\uparrow$ identity matrix

H.S. algebra: basic problem $ax = b \Rightarrow x = a^{-1} b$

$3x = 21 \Rightarrow x = 3^{-1} \cdot 21$

Now we have vector problems of similar form $A\vec{x} = \vec{b}$ (solve for $\vec{x}$)

If guided by H.S. algebra, $\vec{x} = A^{-1}\vec{b}$.

Inverses to matrices:

Have a matrix A

If it has an inverse, C, the properties required of C

$$AC = I = CA$$

Say A has inverse $C = A^{-1}$, then say A is invertible / nonsingular.

A necessary condition on matrix A: A must be square.

(again. this is not sufficient)

Ex.) $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is singular

$$A \underset{\uparrow}{\underline{C}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} ?$$

no such C

Method to find A^{-1}:

- Augment A by the identity I $\qquad [A \mid I]$
- Use EROs to bring this augmented matrix to RREF.
- If, in RREF, the part to the left of the augmentation bar looks like I, then the part to the right is A^{-1}.

Ex.) Say A is 2×2. Follow this process for finding A^{-1}.

General 2×2 $\qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$

Augmented

Case: $a \neq 0$

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \overset{\frac{1}{a}r_1 \to r_1}{\sim} \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ c & d & 0 & 1 \end{array}\right]$$

$$\overset{r_2 - cr_1 \to r_2}{\sim} \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & \frac{ad-bc}{a} & -\frac{c}{a} & 1 \end{array}\right]$$

$$\overset{\frac{a}{ad-bc} r_2 \to r_2}{\sim} \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

(Step requires $ad-bc \neq 0$)

$$\overset{r_1 - \frac{b}{a} r_2 \to r_1}{\sim} \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$\frac{1}{a} - \frac{b}{a} \cdot \frac{-c}{ad-bc} = \frac{ad-bc}{a(ad-bc)} + \frac{bc}{a(ad-bc)}$$

$$= \frac{ad}{a(ad-bc)} = \frac{d}{ad-bc}$$

In the instance where $a \neq 0$, $ad-bc \neq 0$, get

$$A^{-1} = \begin{bmatrix} d/(ad-bc) & -b/(ad-bc) \\ -c/(ad-bc) & a/(ad-bc) \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Whole process for 2×2 matrix hinges on $ad-bc \neq 0$.

Result of our work:

1. $\begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}$ has inverse $\frac{1}{15-2}\begin{bmatrix} 5 & -2 \\ -1 & 3 \end{bmatrix}$

2. $\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$ has no inverse (it's singular)

since $(4)(1) - (-2)(-2) = 0$.

Q: How would you find A^{-1} for

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ -3 & 1 & 0 \end{bmatrix} ?$$

Apply the process $\left[\begin{array}{c|ccc} A & 1 & 0 & 0 \\ & 0 & 1 & 0 \\ & 0 & 0 & 1 \end{array}\right] \hookrightarrow$ RREF $[R \mid C]$

Case: $R = I_3$ — A is nonsingular, and $A^{-1} = C$

Case: $R \neq I_3$ — A is singular (no inverse to A exists)

Q: Why does this process work?

$A\vec{x} = \vec{b}$ solved by $[A \mid \vec{b}] \xhookrightarrow{\text{EROs}} [R \mid \vec{\beta}]$

Note: If we solve

$A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ via GE $\left[\begin{array}{c|c} A & \begin{array}{c} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{array} \end{array}\right] \hookrightarrow$ RREF $[R \mid \vec{\beta}_1]$

first col. of A^{-1}.

$\mathrm{RREF}(A) = I$

$$A \cdot \underset{\begin{bmatrix} \vec{\beta}_1 & \vec{\beta}_2 & \cdots & \vec{\beta}_n \end{bmatrix}}{A^{-1}} = \begin{bmatrix} A\vec{\beta}_1 & A\vec{\beta}_2 & \cdots & A\vec{\beta}_n \end{bmatrix}$$

1st col. of I

2nd col.

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$