Math 231, Thu 11-Feb-2021 -- Thu 11-Feb-2021
Differential Equations and Linear Algebra
Spring 2020

Thursday, February 11th 2021

Topic:: Homogeneous vs. nonhomogeneous problems
Topic:: Null space of a matrix
Read:: ODELA 1.5
HW?? WW 1.05-1.06spanNullspLinInd due Sat. 11 pm
Due:: WW 1.03-04inverses due at 11 pm

Solve problems:
1. [2 -1 4; 3 1 1; -1 2 -5] ~~x~~ b = [-5; 10; 13]
Answer: x = [1; 7; 0] + t[-1; 2; 1]
Note how the two parts behave
A([1; 7; 0] + t[-1; 2; 1]) = A [1; 7; 0] + tA[-1; 2; 1]
= [-5; 10; 13] + [0; 0; 0]
Infinitely many solutions because A has a free column
structure of the set of solutions is a line thru origin + particular vector
all vectors z = t[-1; 2; 1] satisfy Az = 0

2. [1 2 -2 -1; 2 4 1 8; -1 -2 1 -1; 3 6 -1 7] x = [1; 17; -4 18]
Answer: x = [7; 0; 3; 0] + s[-2; 1; 0; 0] + t[-3; 0; -2; 1]

null space of a matrix
- define it
- find it for the two examples above
- Note: 0 vector is always in null(A)
- If A has only pivot columns, then null(A) = {0}, just one vector
- Example: Find null([-2 3 -7 -1 -2; 1 2 7 2 16; 3 -2 13 1 9; 2 2 12 -1 11]
- Whenever Ax = b is consistent,
solutions will take form: (some nonzero vector) + null(A)
when A has no free columns,
Ax = 0 has only the zero solution
Ax = b has as most one solution

One new term:

Did not address

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 - so far
    Is b in the span of set of vectors
    Are there weights such that a linear combination of vectors produces b?
    Does a system of m equations in n unknowns have a solution?
    Is Ax = b consistent?
 - new
    Is b in the column space of A
    Col(A) is same as range of the map f:R^n -> R^m given by f(x) = Ax
    When we solve Ax=b we
      are successful only if b is in this range (Col(A))
      find all x in the domain that "map" to b
      Null(A) consists of those x in domain that map to 0 vector
 - Determine whether any/all are true by doing GE on augmented  [A | b]

Can use GE to describe the column space of A_{mxn}
 - if RREF(A) has a pivot in every row, then col(A) = R^m
 - when RREF(A) has a row of zeros at bottom, the story is more interesting
    example:  A = [2 -1 5; 1 1 1; -1 2 -4]
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Consider the problem:

$$\begin{bmatrix} 2 & -1 & 4 \\ 3 & 1 & 1 \\ -1 & 2 & -5 \end{bmatrix} \vec{x} = \begin{bmatrix} -5 \\ 10 \\ 13 \end{bmatrix}$$

Aug. matrix

$$\left[\begin{array}{ccc|c} 2 & -1 & 4 & -5 \\ 3 & 1 & 1 & 10 \\ -1 & 2 & -5 & 13 \end{array}\right] \sim \begin{array}{c} \overbrace{x_1 \quad x_2}^{\text{pivot}} \quad \overset{\text{free}}{x_3} \\ \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \to \begin{array}{l} x_1 + x_3 = 1 \\ x_2 - 2x_3 = 7 \end{array}$$

$x_3 = t \in \mathbb{R}$

$$\left.\begin{array}{l} x_1 = 1 - x_3 \\ x_2 = 7 + 2x_3 \end{array}\right\} \text{solved for pivot vars.}$$

Solns. to orig. problem

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1-t \\ 7+2t \\ t \end{bmatrix} = \overset{\text{non-t part}}{\begin{bmatrix} 1 \\ 7 \\ 0 \end{bmatrix}} + \overset{\text{t-part}}{t \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}}, \quad t \in \mathbb{R}$$

What is $A \cdot \begin{pmatrix} \text{non-t} \\ \text{part} \end{pmatrix}$?

$$\begin{bmatrix} 2 & -1 & 4 \\ 3 & 1 & 1 \\ -1 & 2 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 7 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} + 7 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} 4 \\ 1 \\ -5 \end{bmatrix}$$

$$= \begin{bmatrix} -5 \\ 10 \\ 13 \end{bmatrix}$$ — desired outcome

What is $A \cdot (\text{t-part})$?

True: ① $A(c\vec{v}) = c(A\vec{v})$
② $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$

$t \cdot \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 2 & -1 & 4 \\ 3 & 1 & 1 \\ -1 & 2 & -5 \end{bmatrix}\begin{bmatrix} -t \\ 2t \\ t \end{bmatrix} = t\begin{bmatrix} 2 & -1 & 4 \\ 3 & 1 & 1 \\ -1 & 2 & -5 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \quad \text{by } ⑦$$

$$= t\left((-1)\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 1 \\ -5 \end{bmatrix}\right) = t\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Our soln. to orig. $A\vec{x} = \begin{bmatrix} -5 \\ 10 \\ 13 \end{bmatrix}$ resulted in a composite answer

$$\underbrace{\begin{bmatrix} 1 \\ 7 \\ 0 \end{bmatrix}}_{\text{single soln. to orig.}} + \underbrace{t\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}}_{\text{when mult. by } A \text{ contributes } \vec{0}}$$

In fact: Had we solved $A\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ in place of $\begin{bmatrix} -5 \\ 10 \\ 13 \end{bmatrix}$ we would have gotten answer:

$$t \cdot \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$ (a line of vectors w/ $\vec{0}$ as one)

Ex.) (Not an isolated instance)

$$\left[\begin{array}{cccc|c} 1 & 2 & -2 & -1 & 1 \\ 2 & 4 & 1 & 8 & 17 \\ -1 & -2 & 1 & -7 & -4 \\ 3 & 6 & -1 & 7 & 18 \end{array}\right] \sim \left[\begin{array}{cccc|c} \boxed{1} & 2 & 0 & 3 & 7 \\ 0 & 0 & \boxed{1} & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

A $\quad \vec{b}$

solving $A\vec{x} = \vec{b}$

Free: x_2, x_4

Set $x_2 = s$, $x_4 = t$

$x_1 = 7 - 2s - 3t$

$x_3 = 3 - 2t$

(plane of vectors w/ $\vec{0}$ included)

Solns.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7-2s-3t \\ s \\ 3-2t \\ t \end{bmatrix} = \underbrace{\begin{bmatrix} 7 \\ 0 \\ 3 \\ 0 \end{bmatrix}}_{A\cdot(\text{this})\text{ hits target}} + \underbrace{s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \end{bmatrix}}_{A\text{ times this gives }\vec{0}.}$$

Note

$$A\begin{bmatrix} 7 \\ 0 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 17 \\ -4 \\ 18 \end{bmatrix}$$

while

$$A\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad A\begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Call the null space of $A_{m\times n}$ the set of vectors $\vec{x}$ satisfying $A\vec{x} = \vec{0}$.

In the previous two examples

1. $\text{Null}(A) = t\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$

   Could have found it directly by solving $A\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

2. $\text{Null} = s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \end{bmatrix}$, i.e., all linear combs. of $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

   Could have found again by solving $A\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ directly.