Math 231, Fri 12-Feb-2021 -- Fri 12-Feb-2021
Differential Equations and Linear Algebra
Spring 2020

Friday, February 12th 2021

Topic:: Column space
Topic:: Linear independence
Read:: ODELA 1.6

$\vec{b}$, collection $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$

One new term:
- so far
  - Is b in the span of set of vectors
  - Are there weights such that a linear combination of vectors produces b?
  - Does a system of m equations in n unknowns have a solution?
  - Is Ax = b consistent?

$$\begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \vec{x} = \vec{b}$$

- new
  - Is b in the column space of A
  - Col(A) is same as range of the map $f: R^n \to R^m$ given by $f(x) = Ax$
  - When we solve Ax=b we
    - are successful only if b is in this range (Col(A))
    - find all x in the domain that "map" to b
    - Null(A) consists of those x in domain that map to 0 vector
- Determine whether any/all are true by doing GE on augmented [A | b]

Can use GE to describe the column space of A_{mxn}
- if RREF(A) has a pivot in every row, then $col(A) = R^m$
- when RREF(A) has a row of zeros at bottom, the story is more interesting
  - example: A = [2 -1 5; 1 1 1; -1 2 -4]

Take a nonzero vector u_1 in R^n
- What would a vector w in R^n look like if it were in $span(u_1)$?
  What would RREF([u_1 w]) look like?
  $span(u_1, w) = span(u_1)$ = line through origin in R^4

- Suppose u_2 is in R^4 and is not in $span(u_1)$.
  Note how this means u_2 goes off in another direction besides u_1
  $span(u_1, u_2)$, a plane, is different from $span(u_1)$, a line

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     u_1, u_2 are linearly independent
   What would RREF([u_1 u_2]) look like?

Do a null space problem
 - make it a matrix with a nontrivial nullspace, perhaps #{rows} > #{cols}
 - Write equivalent form of problem: generating 0 vector as lin.comb. cols of A
    Why does a nontrivial soln exist?
    Geometrically, a nontrivial soln describe nontrivial paths to 0
 - Defn: linear independence
    Same as saying, with vectors at hand, only path to 0 is never to leave
    A statement about a collection
      linear dependence is the opposite
    Test for it
      collection containing just one vector?
      two L.I./L.D. vectors
      RREF as a test
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Given a matrix $A_{m \times n}$ can discuss

- null space — consists of vectors $\vec{x} \in \mathbb{R}^n$ satisfying $A\vec{x} = \vec{0}$
- column space – consists of all vectors $\vec{b}$ (destinations) that make $A\vec{x} = \vec{b}$ consistent

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Ex.) Describe the col(A), where $A = \begin{bmatrix} 2 & -1 & 5 \\ 1 & 1 & 1 \\ -1 & 2 & -4 \end{bmatrix}$

Want to describe possible "destinations" $\vec{b} = \langle b_1, b_2, b_3 \rangle$ – i.e. columns we can augment to A so that the problem $A\vec{x} = \vec{b}$ is consistent.

$$\left[\begin{array}{ccc|c} 2 & -1 & 5 & b_1 \\ 1 & 1 & 1 & b_2 \\ -1 & 2 & -4 & b_3 \end{array}\right] \overset{r_1 \leftrightarrow r_2}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 1 & b_2 \\ 2 & -1 & 5 & b_1 \\ -1 & 2 & -4 & b_3 \end{array}\right]$$

$$\overset{2r_3 + r_2 \to r_2}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 1 & b_2 \\ 0 & 3 & -3 & b_1 + 2b_3 \\ -1 & 2 & -4 & b_3 \end{array}\right]$$

$$\overset{r_1 + r_3 \to r_3}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 1 & b_2 \\ 0 & 3 & -3 & b_1 + 2b_2 \\ 0 & 3 & -3 & b_2 + b_3 \end{array}\right]$$

$$\overset{r_3 - r_2 \to r_3}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 1 & b_2 \\ 0 & 3 & -3 & b_1 + 2b_2 \\ 0 & 0 & 0 & b_3 - b_2 - b_1 \end{array}\right]$$

(row) echelon form

$b_3 = b_1 + b_2$

To be consistent, need $b_3 - b_2 - b_1 = 0$ ← one constraint on the 3 components of $\vec{b}$

So $Col(A)$ has 2 degrees of freedom

$\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ is in $Col(A)$

but $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ isn't.

$$\begin{bmatrix} 7 \\ -20 \\ \underline{-13} \end{bmatrix}$$

Say I have a vector $\vec{v}_1 \in \mathbb{R}^m$

Ex.) $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -3 \\ 1 \end{bmatrix}$

What would a $\vec{w}$ look like if it is in $\text{span}(\vec{v}_1)$?

Perhaps: $\vec{w} = \begin{bmatrix} 4 \\ 2 \\ -6 \\ 2 \end{bmatrix}, \begin{bmatrix} -6 \\ -3 \\ 9 \\ -3 \end{bmatrix}$, etc.

What if we built a matrix from $\vec{v}_1, \vec{w}$ and took it to echelon form?

$\vec{u}_1 \cdot c = \vec{w}$

$$\begin{bmatrix} \vec{u}_1 \downarrow & \vec{w} \downarrow \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & * \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}$$

$\text{span}(\vec{u}_1, \vec{w}) = \text{span}(\vec{u}_1)$

On the other hand, if select $\vec{u}_2$ not in span of $\vec{u}_1$, then building a matrix from $\vec{u}_1, \vec{u}_2$ leads to RREF

$$\begin{bmatrix} \vec{u}_1 & \vec{u}_2 \\ \downarrow & \downarrow \\ & \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} \\ \begin{matrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{matrix} \end{bmatrix}$$

We might say $\vec{u}_1, \vec{u}_2$ are <u>linearly independent</u>.

$\text{span}(\vec{u}_1, \vec{u}_2)$ includes more things than $\text{span}(\vec{u}_1)$

Take $\vec{u}_1, \vec{u}_2$ <u>linearly independent</u> and a third vector $\vec{w}$.

If $\vec{w}$ is in $\text{span}(\vec{u}_1, \vec{u}_2)$:

$$\begin{bmatrix} u_1 & u_2 & w \\ \downarrow & \downarrow & \downarrow \\ & & \\ & & \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}$$

If $\vec{w}$ is not in span of $\vec{u}_1, \vec{u}_2$, then

$$\begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{w} \\ \downarrow & \downarrow & \downarrow \\ & & \\ & & \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}} \\ \begin{matrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \end{matrix} \end{bmatrix}$$

$\text{Span}(\vec{u}_1, \vec{u}_2, \vec{w})$ includes new destinations not found in $\text{span}(\vec{u}_1, \vec{u}_2)$.

Say a collection of vectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n \in \mathbb{R}^m$ is *linearly independent* if (precisely when) the only weights $c_1, c_2, \ldots, c_n$ that produce, under linear combination

$$\vec{0} = c_1 \vec{u}_1 + c_2 \vec{u}_2 + \cdots + c_n \vec{u}_n,$$

the zero vector that work are $c_1 = c_2 = \cdots c_n = 0$.

When $\vec{u}_1, \dots, \vec{u}_n$ are not linearly independent, say they are <u>linearly dependent</u>.

Equivalent notions:

$$\{\vec{u}_1, \dots, \vec{u}_n\} \text{ L.I.} \iff \text{RREF}\begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \\ 0 & \cdots & & 0 \\ 0 & \cdots & & 0 \end{bmatrix}$$

$$\iff \text{Null}\left(\begin{bmatrix} u_1 & u_2 & \cdots & u_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}\right) = \{\vec{0}\}.$$

$$\{\vec{u}_1, \dots, \vec{u}_n\} \text{ L.D.} \iff \begin{bmatrix} \vec{u}_1 & \cdots & \vec{u}_n \\ \downarrow & & \downarrow \end{bmatrix} \text{ has at least one free column}$$

$\iff$ some column $\vec{u}_j$ is in span of the remaining columns.