Math 231, Wed 17-Feb-2021 -- Wed 17-Feb-2021
Differential Equations and Linear Algebra
Spring 2020

Wednesday, February 17th 2021

Note:: Ash Wednesday

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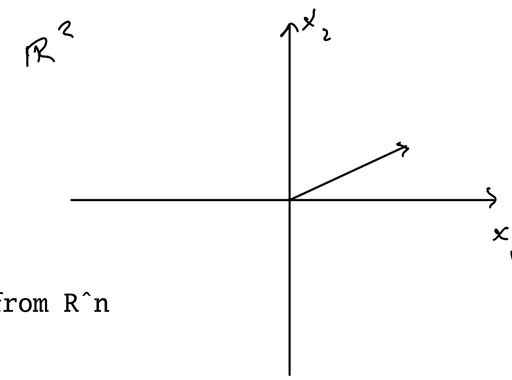
Wk 3, We
Topic:: Bases for null, column spaces
~~Topic:: Determinants~~
Read:: ODELA ~~1.9~~ 1.8 - 1.9
[redacted]

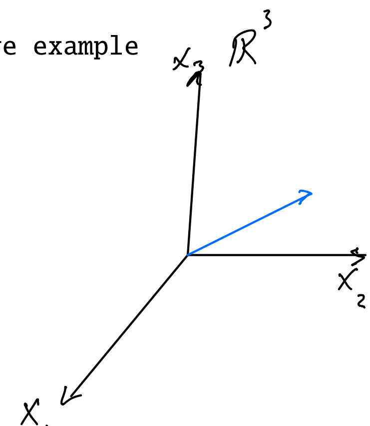


Last time:
 - Said the collections R^n
   are vector spaces, n = 1, 2, ...
   are n-dimensional
     there are many(!) bases for R^n
     every basis of R^n contains exactly n L.I. vectors from R^n
 - Said there are subspaces lying inside R^n

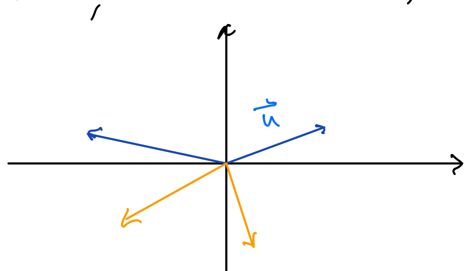
Descriptions of the subspaces of
  R^2: $\mathbb{R}^2$ itself, lines that include the origin (1-dim'l subspaces)
  trivial subspace $\{\vec{0}\}$ (0-dim'l)
  R^3: 1-dim'l subspaces are lines thru the origin
  2-dim'l subspaces are planes thru the origin
  R^8?

$\vec{x} \in \mathbb{R}^2$ has $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Not just any collection of vectors found in R^n is a subspace. Give example



In $\mathbb{R}^2$, take $S = \{ \langle x_1, x_2 \rangle \mid x_2 \geq 0 \}$



Note: $\vec{u} \in S$
but $(-1)\vec{u}$ is not $\in S$

S is not closed under

scalar mult., so it is not a subspace of $\mathbb{R}^2$.

Key feature of a subspace: closed under addition and scalar mult.

Examples of subspace relationships:

1. Given any collection of m-vectors, their span is a subspace of R^m

   ==> If A is m-by-n, Col(A) is a subspace of R^m

$$\text{span}\left(\begin{bmatrix}2\\1\\1\\0\end{bmatrix}, \begin{bmatrix}3\\0\\-1\\2\end{bmatrix}\right) \text{ is a subspace of } \mathbb{R}^4$$

2. Given any m-by-n matrix, Null(A) is a subspace of R^n

Closed under rescaling? $\vec{x} \in \text{null}(A)$ (so $A\vec{x} = 0$), true? $A(t\vec{x}) = \vec{0}$?

" " addition? $\vec{x}_1, \vec{x}_2 \in \text{null}(A)$ (so $A\vec{x}_1 = 0$, $A\vec{x}_2 = \vec{0}$)

true? $A(\vec{x}_1 + \vec{x}_2) = \vec{0}$?

Q1: Can we find bases for Null(A) and Col(A)?

Note: The columns of A "span" the column space Col(A).

Finding a basis for Col(A) is a matter of "pairing down" to a L.I. collection.

Example: A = [2 4 3 1 8; 3 6 2 4 7; -1 -2 2 -4 3; 2 4 -1 5 0]

$$A = \begin{bmatrix} 2 & 4 & 3 & 1 & 8 \\ 3 & 6 & 2 & 4 & 7 \\ -1 & -2 & 2 & -4 & 3 \\ 2 & 4 & -1 & 5 & 0 \end{bmatrix}$$
$$\begin{matrix} p & f & p & f & f \end{matrix}$$

Col(A) is spanned by $\underset{\vec{u}_1}{\begin{bmatrix}2\\3\\-1\\2\end{bmatrix}}, \begin{bmatrix}4\\6\\-2\\4\end{bmatrix}, \underset{\vec{u}_2}{\begin{bmatrix}3\\2\\2\\-1\end{bmatrix}}, \begin{bmatrix}1\\4\\-4\\5\end{bmatrix}, \begin{bmatrix}8\\7\\3\\0\end{bmatrix}$

Basis for Col(A): $\vec{u}_1, \vec{u}_2$

Define: rank(A) and nullity(A).

$$\left.\begin{array}{l} \text{rank}(A) = \#\text{ of pivot cols in } A \\ \text{nullity}(A) = \#\text{ of free cols in } A \end{array}\right\} \text{as revealed by RREF}$$

Example: Having found RREF(A), is there another basis for Col(A) we can see?

Example: (a) Find a basis for the collection of vectors
<s - 2t, 3s + 2w, s + t + w, t-3w>

(b) We called what we found a basis, which presumes this collection is a subspace of something. What larger space does it reside in? How do we know it is a subspace?

(c) Can you write a matrix A whose column space corresponds to this collection of vectors?

(d) Can our basis be "enhanced" in order to create a basis for R^4?

Q2: Suppose b is a nonzero vector, and Ax = b is consistent.
Do the solutions of Ax = b form a subspace of R^n?

Q3: (If there is time)
Visit the website https://pad.disroot.org/p/m231-17feb2021
and write, as a class, things we can conclude in each setting.

To consider

- linear independence of functions on an interval
  - 1, $\sin^2 x$, $\cos^2 x$ are L.D.
  - specification of interval is important!
    - Example: x and |x| on $(0,\infty)$ vs. $(-\infty,\infty)$
  - Test:
    - Form n-by-n matrix, fns along top row, derivs. up to order (n-1) down.
    - If at some $t \in I$, A(t) has no free col., then fns are L.I. on I.

Fact: ① $\dim(\mathrm{Col}(A)) = \mathrm{rank}(A)$

② $\dim(\mathrm{Null}(A)) = \mathrm{nullity}(A)$

$$A = \left[\begin{array}{ccccc|c} 2 & 4 & 3 & 1 & 8 & 0 \\ 3 & 6 & 2 & 4 & 7 & 0 \\ -1 & -2 & 2 & -4 & 3 & 0 \\ 2 & 4 & -1 & 5 & 0 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Null(A) involves solving $A\vec{x} = \vec{0}$

In RREF

Row 1 says $x_1 + 2x_2 + 2x_4 + x_5 = 0$

2 says $x_3 - x_4 + 2x_5 = 0$

Free vars: x_2, x_4, x_5

$$x_1 = -2x_2 - 2x_4 - x_5$$

$$x_3 = x_4 - 2x_5$$

Vectors in Null(A):

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 2x_4 - x_5 \\ x_2 \\ x_4 - 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

These 3 vectors form a basis of Null(A)