Math 231, Thu 18-Feb-2021 -- Thu 18-Feb-2021
Differential Equations and Linear Algebra
Spring 2020

Thursday, February 18th 2021

Due::   WW LAConcepts1.03-1.08 due at 11 pm

Other calendar items

Thursday, February 18th 2021

Wk 3, Th
Topic:: Determinants
Topic:: Cramer’s rule
Read:: ODELA 1.9-1.10

https://pad.disroot.org/p/m231-17feb2021

Exercise:

(a) Find a basis for the collection of vectors
<s - 2t, 3s + 2w, s + t + w, t-3w>

$$\begin{bmatrix} s-2t \\ 3s+2w \\ s+t+w \\ t-3w \end{bmatrix} = s\begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix} + w\begin{bmatrix} 0 \\ 2 \\ 1 \\ -3 \end{bmatrix}$$

$s, t, w \in \mathbb{R}$

basis:

$$\begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ -3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} - & - & - & b_1 \\ - & - & - & b_2 \\ - & - & - & b_3 \\ - & - & - & b_4 \end{array}\right]$$

(b) We called what we found a basis, which presumes this collection is a subspace of something. What larger space does it reside in? How do we know it is a subspace?

$\mathbb{R}^4$

(c) Can you write a matrix A whose column space corresponds to this collection of vectors?

(d) Can our basis be "enhanced" in order to create a basis for R^4?

Q2: Suppose b is a nonzero vector, and Ax = b is consistent. Do the solutions of Ax = b form a subspace of R^n?

Q3: (If there is time)
Visit the website https://pad.disroot.org/p/m231-17feb2021
and write, as a class, things we can conclude in each setting.

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To consider
 - linear independence of functions on an interval
    1, sin^2 x, cos^2 x  are L.D.
    specification of interval is important!
      Example: x and |x| on (0,\infty) vs. (-\infty,\infty)
    Test:
      Form n-by-n matrix, fns along top row, derivs. up to order (n-1) down.
      If at some t\in I, A(t) has no free col., then fns are L.I. on I.

Determinants
 - 2-by-2:  | a   b |
            | c   d | = ad - bc
   indicates a singular matrix when zero (parallel lines)
 - What a determinant can determine
 - notation |A|, det(A)
 - extending to other square matrices using cofactor expansion
    recursive definition
    same result whether you expand along one row/col or another
    may lead you to choose row/col with most zeros
 - Theorem: Determinant of a triangular A is product of its diagonal elements.
 - Theorem: Determinants and EROs. If B arises from A due to
     a row swap, then |B| = -|A|.
     a row of B is r times a row of A, then |B| = r|A|.
     a row of B is one row of A plus r times another row of A, then |B| = |A|.
 - Theorem: For n-by-n matrix A, TFAE:
     A is nonsingular
     det(A) \ne 0.
     cols of A form a basis for R^n
     RREF(A) = I
     Every b of R^n is in col(A)
     null(A) is trivial
     rank(A) = n
     nullity(A) = 0
 - Cramer's Rule
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Cramer's Rule

Cramer's rule provides a method for solving a system of linear algebraic equations for which the associated matrix problem $\mathbf{Ax} = \mathbf{b}$ has a coefficient matrix which is *nonsingular*. It is of no use if this criterion is not met and, considering the effectiveness of algorithms we have learned already for solving such a system (inversion of the matrix $\mathbf{A}$, and Gaussian elimination, specifically), it is not clear why we need yet another method. Nevertheless, it is a tool (some) people use, and should be recognized/understood by you when you run across it. We will describe the method, but not explain why it works, as this would require a better understanding of determinants than our time affords.

So, let us assume the n-by-n matrix $\mathbf{A}$ is nonsingular, that $\mathbf{b}$ is a known vector in $\mathbb{R}^n$, and that we wish to solve the equation $\mathbf{Ax} = \mathbf{b}$ for an unknown (unique) vector $\mathbf{x} \in \mathbb{R}^n$. Cramer's rule requires the construction of matrices $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$, where each $\mathbf{A}_j$, $1 \leqslant j \leqslant n$ is built from the original $\mathbf{A}$ and $\mathbf{b}$. These are constructed as follows: the j^{th} column of $\mathbf{A}$ is replaced by $\mathbf{b}$ to form $\mathbf{A}_j$.

Example 1: Construction of $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ when $\mathbf{A}$ is 3-by-3

Suppose $\mathbf{A} = (a_{ij})$ is a 3-by-3 matrix, and $\mathbf{b} = (b_i)$, then

$$\mathbf{A}_1 = \begin{pmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{pmatrix}, \quad \mathbf{A}_2 = \begin{pmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{pmatrix}, \quad \text{and} \quad \mathbf{A}_3 = \begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{pmatrix}.$$

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Armed with these $\mathbf{A}_j$, $1 \leqslant j \leqslant n$, the solution vector $\mathbf{x} = (x_1, \ldots, x_n)$ has its j^{th} component given by

$$x_j = \frac{|\mathbf{A}_j|}{|\mathbf{A}|}, \qquad j = 1, 2, \ldots, n. \tag{1}$$

It should be clear from this formula why it is necessary that $\mathbf{A}$ be nonsingular.

Example 2:

Use Cramer's rule to solve the system of equations

$$\begin{aligned} x + 3y + z - w &= -9 \\ 2x + y - 3z + 2w &= 51 \\ x + 4y + 2w &= 31 \\ -x + y + z - 3w &= -43 \end{aligned}$$

Here, $\mathbf{A}$ and $\mathbf{b}$ are given by

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 1 & -1 \\ 2 & 1 & -3 & 2 \\ 1 & 4 & 0 & 2 \\ -1 & 1 & 1 & -3 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} -9 \\ 51 \\ 31 \\ -43 \end{pmatrix}, \qquad \text{so} \qquad |\mathbf{A}| = \begin{vmatrix} 1 & 3 & 1 & -1 \\ 2 & 1 & -3 & 2 \\ 1 & 4 & 0 & 2 \\ -1 & 1 & 1 & -3 \end{vmatrix} = -46.$$

Thus,

$$x = \frac{|\mathbf{A}_1|}{|\mathbf{A}|} = \frac{1}{|\mathbf{A}|}\begin{vmatrix} -9 & 3 & 1 & -1 \\ 51 & 1 & -3 & 2 \\ 31 & 4 & 0 & 2 \\ -43 & 1 & 1 & -3 \end{vmatrix} = \frac{-230}{-46} = 5,$$

$$y = \frac{|\mathbf{A}_2|}{|\mathbf{A}|} = \frac{1}{|\mathbf{A}|}\begin{vmatrix} 1 & -9 & 1 & -1 \\ 2 & 51 & -3 & 2 \\ 1 & 31 & 0 & 2 \\ -1 & -43 & 1 & -3 \end{vmatrix} = \frac{-46}{-46} = 1,$$

$$z = \frac{|\mathbf{A}_3|}{|\mathbf{A}|} = \frac{1}{|\mathbf{A}|}\begin{vmatrix} 1 & 3 & -9 & -1 \\ 2 & 1 & 51 & 2 \\ 1 & 4 & 31 & 2 \\ -1 & 1 & -43 & -3 \end{vmatrix} = \frac{276}{-46} = -6,$$

$$w = \frac{|\mathbf{A}_4|}{|\mathbf{A}|} = \frac{1}{|\mathbf{A}|}\begin{vmatrix} 1 & 3 & 1 & -9 \\ 2 & 1 & -3 & 51 \\ 1 & 4 & 0 & 31 \\ -1 & 1 & 1 & -43 \end{vmatrix} = \frac{-506}{-46} = 11,$$

yielding the solution $\mathbf{x} = (x, y, z, w) = (5, 1, -6, 11)$.

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Determinants

- Know how for a 2-by-2 $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$: $\quad \det = ad - bc$
- Notation

  $\det(A)$, $\quad |A|$
- Use: determines whether a matrix singular or not

  0 determinant $\longleftrightarrow$ singular = no inverse

  nonzero determinant $\longleftrightarrow$ nonsingular = invertible
- $\det(A)$ makes sense only for square matrices

Example:

4x4 matrix A det.

$$\begin{vmatrix} 2 & 1 & 0 & 3 \\ 1 & -1 & 4 & 0 \\ 2 & 1 & 0 & -1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = \underbrace{(1)(-1)^{4+1}\begin{vmatrix} 1 & 0 & 3 \\ -1 & 4 & 0 \\ 1 & 0 & -1 \end{vmatrix}}_{C_{4,1}} + \underbrace{(1)(-1)^{4+2}\begin{vmatrix} 2 & 0 & 3 \\ 1 & 4 & 0 \\ 2 & 0 & -1 \end{vmatrix}}_{C_{4,2}}$$

$$+ \underbrace{(1)(-1)^{4+3}\begin{vmatrix} 2 & 1 & 3 \\ 1 & -1 & 0 \\ 2 & 1 & -1 \end{vmatrix}}_{C_{4,3}} + \underbrace{(0)(-1)^{4+4}\begin{vmatrix} 2 & 1 & 0 \\ 1 & -1 & 4 \\ 2 & 1 & 0 \end{vmatrix}}_{C_{4,4}}$$

pos. 4,1 $\quad$ pos. 4,2 $\quad$ pos. 4,3 $\quad$ pos. 4,4

$C_{i,j}$ called (i,j)-cofactor of matrix (A).

We have, above, "expanded our $\det(A)$ in cofactors along 4^{th} row"

that is,

$$\det(A) = \underline{a_{41}C_{41} + a_{42}C_{42} + a_{43}C_{43} + a_{44}C_{44}}$$

Surprising (?)

Expansion of $\det(A)$ in cofactors along

- any row or
- any column

always gives the same final number/result.

Given this, expanding along col. 3

$$\det(A) = a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} + a_{34}C_{34}$$

$$= \cancel{0(-1)^{1+3}\begin{vmatrix} 1 & -1 & 0 \\ 2 & 1 & -1 \\ 1 & 1 & 0 \end{vmatrix}} + 4(-1)^{5}\begin{vmatrix} 2 & 1 & 3 \\ 2 & 1 & -1 \\ 1 & 1 & 0 \end{vmatrix} + (1)(-1)^{7}\begin{vmatrix} 2 & 1 & 3 \\ 1 & -1 & 0 \\ 2 & 1 & -1 \end{vmatrix}$$

$$\begin{vmatrix} 2 & 1 & 3 \\ 2 & 1 & -1 \\ 1 & 1 & 0 \end{vmatrix} = 1(-1)^{3+1}\begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix} + (1)(-1)^{3+2}\begin{vmatrix} 2 & 3 \\ 2 & -1 \end{vmatrix}$$

$$= (1)(1)(-4) + (1)(-1)(-8) = \underline{4}$$

$$\begin{vmatrix} 2 & 1 & \underline{3} \\ 1 & -1 & \underline{0} \\ 2 & 1 & \underline{-1} \end{vmatrix} = (3)(-1)^{1+3} \underline{\underline{\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}}} + 0 + (-1)(-1)^{3+3} \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= (3)(1)(3) - (-3) = 12$$

Original

$$\det(A)_{4\times4} = (-4)(4) - 12 = -28$$

$$A = \begin{bmatrix} -5 & 3 \\ -6 & 2 \end{bmatrix}$$

$$[A \mid I] = \left[\begin{array}{cc|cc} -5 & 3 & 1 & 0 \\ -6 & 2 & 0 & 1 \end{array}\right] \underset{\sim}{\xrightarrow{r_1 - r_2 \to r_1}} \left[\begin{array}{cc|cc} 1 & 1 & 1 & -1 \\ -6 & 2 & 0 & 1 \end{array}\right]$$

$$\underset{\sim}{\xrightarrow{r_2 + 6r_1 \to r_2}} \left[\begin{array}{cc|cc} 1 & 1 & 1 & -1 \\ 0 & 8 & 6 & -5 \end{array}\right] \underset{\sim}{\xrightarrow{\frac{1}{8}r_2 \to r_2}} \left[\begin{array}{cc|cc} 1 & 1 & 1 & -1 \\ 0 & 1 & 3/4 & -5/8 \end{array}\right]$$

$$\underset{\sim}{\xrightarrow{r_1 - r_2 \to r_1}} \left[\begin{array}{cc|cc} 1 & 0 & 1/4 & -3/8 \\ 0 & 1 & 3/4 & -5/8 \end{array}\right]$$

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 4 \\ -2 & 5 \end{bmatrix}$$

1. Which $\vec{b} \in \mathbb{R}^3$ make the vector eqn. $A\vec{x} = \vec{b}$ consistent
2. Which $\vec{b}$ can be written as linear combination of $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$

— that is, so $\vec{b} = x_1 \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$

3. Describe the Col(A).

All are asking the same thing, using different language.

$$A = \begin{bmatrix} 4 & 2 & 1 & -2 & -1 \\ 2 & 1 & 3 & -2 & 3 \\ -1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

Col(A) is spanned by $\begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$