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Math 231, Fri 19-Feb-2021 -- Fri 19-Feb-2021
Differential Equations and Linear Algebra
Spring 2020


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Friday, February 19th 2021
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Wk 3, Fr
Topic:: Cramer's rule
Topic:: Eigenvalues and eigenvectors
Read::  ODELA 1.11-1.12
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Determinants

Discuss, for systems of 2 linear (algebraic) equations in 2 unknowns, such as

$$\begin{aligned} ax + by &= e, \\ cx + dy &= f, \end{aligned} \tag{1}$$

- the different solution cases: a unique point of intersection, coincident lines, parallel lines.
- distinguishing the unique solution case from the other two cases based on the ratios $a : c$ and $b : d$ or, better yet, the value of $ad - bc$.
- the form of the associated matrix problem

  $$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix},$$

  and how the quantity $(ad - bc)$ above is a feature of the **coefficient matrix**. For

  $$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad \text{define} \qquad \det(\mathbf{A}) = |\mathbf{A}| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} := ad - bc,$$

  called the **determinant** of $\mathbf{A}$.

  Note that,

  - every time $\mathbf{A}$ is nonsingular, the matrix problem $\mathbf{Ax} = \mathbf{b}$ has a unique solution, and
  - every time $\det(\mathbf{A}) \neq 0$, the matrix problem $\mathbf{Ax} = \mathbf{b}$ has a unique solution.

so the next result should not be so terribly surprising.

> **Theorem 1:** The 2-by-2 matrix
> $$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
> is nonsingular if and only if $\det(\mathbf{A}) \neq 0$.

Q: Is there a way to define $\det(\mathbf{A})$ for n-by-n (square) matrices with $n > 2$ so that this theorem holds when "2-by-2" is replaced by "n-by-n"?
A: Yes.

Note: There is no need to define $\det(\mathbf{A})$ when $\mathbf{A}$ is non-square.

For an n-by-n matrix $\mathbf{A}$, define

- (i, j)-minor of $\mathbf{A}$ to be the determinant of the submatrix of $\mathbf{A}$ which is missing the i^{th} row and j^{th} column of $\mathbf{A}$.

  Note: there are n^2 such minors, denoted M_{ij}, for $1 \leqslant i, j \leqslant n$.

- (i, j)-cofactor of $\mathbf{A}$, denoted C_{ij}, and given by
  $$C_{ij} := (-1)^{i+j} M_{ij}.$$

- determinant of $\mathbf{A}$, given by **cofactor expansion** along the i^{th} row
  $$\det(\mathbf{A}) := a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = \sum_{k=1}^{n} a_{ik}C_{ik},$$
  or by cofactor expansion along the j^{th} column
  $$\det(\mathbf{A}) := a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} = \sum_{k=1}^{n} a_{kj}C_{kj}.$$

  It may seem we have just given $(2n)$ different formulas for $\det(\mathbf{A})$, but each one of them yields the exact same answer. With such freedom, one generally chooses to expand along the row or column that contains the most zero entries.

  Point out the recursive nature of this definition.

Additional facts about determinants:

1. Two $n \times n$ matrices A, B

$$|AB| = |A| \cdot |B|$$

2. Under EROs:

- row swap causes a change only in sign of det.
- rescaling a row by factor c causes the det. to be rescaled by c

Ex.) $A = \begin{bmatrix} 2 & 1 \\ 3 & 7 \end{bmatrix}$ has $\det(A) = 11$

So $\begin{bmatrix} 10 & 5 \\ 3 & 7 \end{bmatrix}$ has $\det = 5 \cdot 11 = 55$

- adding a multiple of one row to another affects no change to the determinant.

3. If A is upper-triangular $\begin{bmatrix} a_{11} & & \\ & a_{22} & \\ \text{zeros below} & & \ddots \; a_{nn} \end{bmatrix}$

then $|A| = a_{11} a_{22} \cdots a_{nn}$.

main diagonal - from top left entry to bottom right

Cramer's rule : One use for determinants
- give context
- do an example (2-by-2?)

Purpose of CR is solve the vector problem $A\vec{x} = \vec{b}$ in the case where

$\vec{x} = A^{-1}\vec{b}$

- A is square and
- A is nonsingular.

Ex. $2x + y = 7$
$x - 3y = 1$

in ~~matrix~~ vector form

$$\underbrace{\begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 7 \\ 1 \end{bmatrix}}_{\vec{b}}$$

Cramer's rule gives formulas for components x, y of $\vec{x} = \langle x, y \rangle$.

$$x = \frac{\begin{vmatrix} 7 & 1 \\ 1 & -3 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 1 & -3 \end{vmatrix}} = \frac{7(-3) - 1}{2(-3) - 1} = \frac{-22}{-7} = \frac{22}{7}.$$

$$y = \frac{\begin{vmatrix} 2 & 7 \\ 1 & 1 \end{vmatrix}}{\det(A)} = \frac{2 - 7}{-7} = \frac{5}{7}.$$

$\vec{v}$ is e-vec of $A_{n\times n}$ when $\vec{v} \neq \vec{0}$ and $A\vec{v} = \lambda\vec{v}$ for some scalar λ.

$$A\vec{v} = \lambda\vec{v} \iff A\vec{v} - \lambda\vec{v} = \vec{0}$$
$$\iff A\vec{v} - \lambda I\vec{v} = \vec{0}$$
$$\iff (A - \lambda I)\vec{v} = \vec{0}$$
$$\iff \vec{v} \text{ is a nonzero vector in } \operatorname{null}(A - \lambda I).$$

Note: $\operatorname{null}(A-\lambda I)$ is a subspace of $\mathbb{R}^n$ with dimension $=$ nullity$(A-\lambda I)$.

Finding e-vals and e-vecs.

1. Must find choices of scalar λ for which

$$\text{nullity}(A - \lambda I) > 0 \iff \boxed{\det(A - \lambda I) = 0}$$

Solve this for values of λ.

Both represent the fact that $A-\lambda I$ is singular

2. For each soln. λ coming from 1, we find $\operatorname{null}(A - \lambda I)$.

Cramer's Rule

Cramer's rule provides a method for solving a system of linear algebraic equations for which the associated matrix problem $\mathbf{Ax} = \mathbf{b}$ has a coefficient matrix which is *nonsingular*. It is of no use if this criterion is not met and, considering the effectiveness of algorithms we have learned already for solving such a system (inversion of the matrix $\mathbf{A}$, and Gaussian elimination, specifically), it is not clear why we need yet another method. Nevertheless, it is a tool (some) people use, and should be recognized/understood by you when you run across it. We will describe the method, but not explain why it works, as this would require a better understanding of determinants than our time affords.

So, let us assume the n-by-n matrix $\mathbf{A}$ is nonsingular, that $\mathbf{b}$ is a known vector in $\mathbb{R}^n$, and that we wish to solve the equation $\mathbf{Ax} = \mathbf{b}$ for an unknown (unique) vector $\mathbf{x} \in \mathbb{R}^n$. Cramer's rule requires the construction of matrices $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$, where each $\mathbf{A}_j$, $1 \leqslant j \leqslant n$ is built from the original $\mathbf{A}$ and $\mathbf{b}$. These are constructed as follows: the j^{th} column of $\mathbf{A}$ is replaced by $\mathbf{b}$ to form $\mathbf{A}_j$.

Example 1: Construction of $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ when $\mathbf{A}$ is 3-by-3

Suppose $\mathbf{A} = (a_{ij})$ is a 3-by-3 matrix, and $\mathbf{b} = (b_i)$, then

$$\mathbf{A}_1 = \begin{pmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{pmatrix}, \quad \mathbf{A}_2 = \begin{pmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{pmatrix}, \quad \text{and} \quad \mathbf{A}_3 = \begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{pmatrix}.$$

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Armed with these $\mathbf{A}_j$, $1 \leqslant j \leqslant n$, the solution vector $\mathbf{x} = (x_1, \ldots, x_n)$ has its j^{th} component given by

$$x_j = \frac{|\mathbf{A}_j|}{|\mathbf{A}|}, \qquad j = 1, 2, \ldots, n. \tag{2}$$

It should be clear from this formula why it is necessary that $\mathbf{A}$ be nonsingular.

Example 2:

Use Cramer's rule to solve the system of equations

$$\begin{aligned} x + 3y + z - w &= -9 \\ 2x + y - 3z + 2w &= 51 \\ x + 4y + 2w &= 31 \\ -x + y + z - 3w &= -43 \end{aligned}$$

$\underline{2}w + \underline{4}y + 1x = 31$

Here, **A** and **b** are given by

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 1 & -1 \\ 2 & 1 & -3 & 2 \\ 1 & 4 & 0 & 2 \\ -1 & 1 & 1 & -3 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} -9 \\ 51 \\ 31 \\ -43 \end{pmatrix}, \qquad \text{so} \qquad |\mathbf{A}| = \begin{vmatrix} 1 & 3 & 1 & -1 \\ 2 & 1 & -3 & 2 \\ 1 & 4 & 0 & 2 \\ -1 & 1 & 1 & -3 \end{vmatrix} = -46.$$

x

Thus,

$$x = \frac{|\mathbf{A}_1|}{|\mathbf{A}|} = \frac{1}{|\mathbf{A}|}\begin{vmatrix} -9 & 3 & 1 & -1 \\ 51 & 1 & -3 & 2 \\ 31 & 4 & 0 & 2 \\ -43 & 1 & 1 & -3 \end{vmatrix} = \frac{-230}{-46} = 5,$$

$$y = \frac{|\mathbf{A}_2|}{|\mathbf{A}|} = \frac{1}{|\mathbf{A}|}\begin{vmatrix} 1 & -9 & 1 & -1 \\ 2 & 51 & -3 & 2 \\ 1 & 31 & 0 & 2 \\ -1 & -43 & 1 & -3 \end{vmatrix} = \frac{-46}{-46} = 1,$$

$$z = \frac{|\mathbf{A}_3|}{|\mathbf{A}|} = \frac{1}{|\mathbf{A}|}\begin{vmatrix} 1 & 3 & -9 & -1 \\ 2 & 1 & 51 & 2 \\ 1 & 4 & 31 & 2 \\ -1 & 1 & -43 & -3 \end{vmatrix} = \frac{276}{-46} = -6,$$

$$w = \frac{|\mathbf{A}_4|}{|\mathbf{A}|} = \frac{1}{|\mathbf{A}|}\begin{vmatrix} 1 & 3 & 1 & -9 \\ 2 & 1 & -3 & 51 \\ 1 & 4 & 0 & 31 \\ -1 & 1 & 1 & -43 \end{vmatrix} = \frac{-506}{-46} = 11,$$

yielding the solution $\mathbf{x} = (x, y, z, w) = (5, 1, -6, 11)$.

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A square (n×n matrix) induces a function from $\mathbb{R}^n$ into $\mathbb{R}^n$

$f(\vec{x}) = A\vec{x}$ yielding output, another vector in $\mathbb{R}^n$

$\vec{x} \in \mathbb{R}^n$

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Eigenvalues and eigenvectors
 - mapping x -> Ax has input and output in R^n when A is square
 - look at animation linked from class website
    try? changing A to [-14 -42; 4 12]   (singular matrix?)
 - Q: When are Ax and x parallel?
    One obvious, but uninteresting answer, is when x=0.
    Revised Q: For which nonzero vectors x are Ax and x parallel?
    Write as Ax = lambda x  <==>  (A - lambda I) x = 0
                            <==>  (A - lambda I) has a nontrivial null space
                            <==>  |A - lambda I| = 0
 - Examples of finding them
    find eigenvalue first
      roots of characteristic polynomial  |A - lambda I|
      degree of characterisic polynomial matches number of rows/cols of A
      quadratic formula vs. factoring
    eigenspaces
      one for each eigenvalue
      another word for a null space, so is a subspace of R^n
      dim(eigenspace) = nullity(A - lambda I)
      know all the corresp. eigenvectors once you have a basis for it
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Square A, $\vec{v}$ is an eigenvector if $A\vec{v} = \lambda\vec{v}$ and $\vec{v} \neq \vec{0}$.

corresponding eigenvalue (to $\vec{v}$)

Eigenvalues and Eigenvectors

Converting n^{th} order DEs and systems of DEs into 1st order systems

Can ignore this for now

- Reminder of how this is done

  for each dependent variable u whose highest appearing derivative is k^{th} order, introduce $(k-1)$ new dependent variables to rename $u', u'', \ldots, u^{(k-1)}$

- When the original DE (or system of DEs) is
  - linear, converted system will be (in the form of) $\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{b}(t)$.
  - linear with constant coefficients, converted system will be $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}(t)$.
  - homogeneous linear with constant coefficients, converted system will be $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Example 3: