From last time: an $n \times n$ matrix $A$ has eigenvector $\vec{v} \in \mathbb{R}^n$ when

i) $\vec{v} \neq \vec{0}$

ii) $A\vec{v} = \lambda \vec{v}$ for some scalar $\lambda$ (which may be 0).

$\uparrow$ corresponding eigenvalue

$(\lambda, \vec{v})$ form an eigenpair

Q1: Given $A = \begin{bmatrix} 7 & -6 \\ 4 & -3 \end{bmatrix}$, is

(a) $\vec{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ an e-vec of $A$? No

(b) $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ an e-vec of $A$? Yes

(c) Is 2 an e-val of $A$? No

(d) Is 3 an e-val of $A$? Yes

(a) $\begin{bmatrix} 7 & -6 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 7 \\ 4 \end{bmatrix} + \begin{bmatrix} -6 \\ -3 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \end{bmatrix}$ X

(b) $\begin{bmatrix} 7 & -6 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix} + \begin{bmatrix} -6 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

✓ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an e-vector, w/ e-val $\underline{1}$.

Friday: $A\vec{v} = \lambda\vec{v} \iff (A - \lambda I)\vec{v} = \vec{0}$

$\iff \vec{v}$ is in $\text{Null}(A - \lambda I)$

For all choices of scalar $\lambda$, $\vec{0}$ is in $\text{Null}(A - \lambda I)$.

It's only special values of $\lambda$ for which $\text{Null}(A - \lambda I)$ is nontrivial (i.e. has something more than just $\vec{0}$). Marker of those $\lambda$: $\det(A - \lambda I) = 0$

(c) See if $\det(A - \lambda I) = 0$ when $\lambda$ is set to 2

$$\left| \begin{bmatrix} 7 & -6 \\ 4 & -3 \end{bmatrix} - \underset{\lambda}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} 5 & -6 \\ 4 & -5 \end{vmatrix} = (5)(-5) - (4)(-6) = -1$$

(d) $$\left| \begin{bmatrix} 7 & -6 \\ 4 & -3 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} 4 & -6 \\ 4 & -6 \end{vmatrix} = 0$$

Math 231, Mon 22-Feb-2021 -- Mon 22-Feb-2021
Differential Equations and Linear Algebra
Spring 2020

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Monday, February 22nd 2021
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Wk 4, Mo
Topic:: Eigenvalues and eigenvectors
Read:: ODELA 1.11-1.12

- Examples: Find eigenpairs for
  1. A = [7/3 4/3; 2/3 5/3]   (from a class example)
  2. A = [2 1; 0 3]
  3. A = [-1 4; 2 -3]
  4. A = [28 100; -9 -32]   (has repeated e-val with GM < AM)
     use the terms: eigenspace, basis for eigenspace
  5. A = [1 2; 3 4]   (has irrational evals)
  6. A = [-1 4 0; 2 -3 0; 1 0 2]

- Questions that do not require all the steps we’ve done
  Is 2 an eigenvalue of [7 -6; 4 -3]? How about 3?
  Is [2; 1] an eigenvector of [7 -6; 4 -3]? How about [1; 1]?
  Given that [1;1;0] is an e-vector of [4 -2 -5; 5 -3 -5; 2 -2 -3], find eval

- Easy to find e-values for triangular matrix
  works for [1 3 0; 0 2 -1; 0 0 5]
  doesn’t work for [0 0 1; 0 2 -1; 3 0 5]

More e-vals/evecs

- Examples:
  A = [1 2; 3 4]   e-vals are $5/2 \pm \sqrt{33}/2$
  A = [1 3; 3 1]   e-vals are -2, 4
  Note: here evecs form orthogonal basis of $R^2$
  A = [1 4; -4 1]   e-vals are $1 \pm 4i$
  Note: evals/evecs come in complex conjugate pairs

Ex.)

$$A = \begin{bmatrix} 7/3 & 4/3 \\ 2/3 & 5/3 \end{bmatrix}$$

Find e-vals and corresp. e-vectors for $A$.

For e-vals, need $\det(A - \lambda I) = 0$.

$$0 = \left| \begin{bmatrix} 7/3 & 4/3 \\ 2/3 & 5/3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} 7/3 - \lambda & 4/3 \\ 2/3 & 5/3 - \lambda \end{vmatrix}$$

$$= \left(\frac{7}{3} - \lambda\right)\left(\frac{5}{3} - \lambda\right) - \left(\frac{2}{3}\right)\left(\frac{4}{3}\right) = \lambda^2 - 4\lambda + \frac{35}{9} - \frac{8}{9} = \underbrace{\lambda^2 - 4\lambda + 3}_{\text{quadratic polynomial}}$$

$0 = (\lambda - 3)(\lambda - 1)$ true only when $\underbrace{\lambda = 1, 3}_{\text{Full list of e-vals.}}$

To get corresp. e-vecs to $\lambda = 1$

Find/describe $\text{Null}(A - 1 \cdot I)$ — solve $\underbrace{\begin{bmatrix} 4/3 & 4/3 \\ 2/3 & 2/3 \end{bmatrix}}_{A - I} \vec{v} = \vec{0}$

GE $\left[\begin{array}{cc|c} 4/3 & 4/3 & 0 \\ 2/3 & 2/3 & 0 \end{array}\right] \xleftrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow v_1 + v_2 = 0$

$v_1 = -v_2$

(free)

Any e-vector corresp. to $\lambda = 1$ looks like

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -v_2 \\ v_2 \end{bmatrix} = v_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

That is, all e-vecs paired w/ $\lambda = 1$ are scalar multiples of $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

— $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ serves as a basis vector for the collection of e-vecs. corresp. to $\lambda = 1$.

(call this the eigenspace corresp. to e-val. $\lambda = 1$)

Similarly, for $\lambda = 3$

$$A - 3I = \begin{bmatrix} 7/3 - 3 & 4/3 \\ 2/3 & 5/3 - 3 \end{bmatrix} = \begin{bmatrix} -2/3 & 4/3 \\ 2/3 & -4/3 \end{bmatrix}$$

So, the eigenspace corresp. to $\lambda = 3$ (same as $\mathrm{Null}(A - 3I)$)

Solve

$$\begin{bmatrix} -2/3 & 4/3 \\ 2/3 & -4/3 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -\frac{2}{3} & 4/3 & 0 \\ 2/3 & 4/3 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$v_1 - 2v_2 = 0$

$v_1 = 2v_2$ free

E-vectors

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2v_2 \\ v_2 \end{bmatrix} = v_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad v_2 \in \mathbb{R}$$

- that is, $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is a basis vector for this eigenspace.