Math 231, Wed 24-Feb-2021 -- Wed 24-Feb-2021
Differential Equations and Linear Algebra
Spring 2020

Wednesday, February 24th 2021

Due:: HC02 at 11 pm

Other calendar items

Wednesday, February 24th 2021

Topic:: Wronskian and L.I. Functions

$$\begin{matrix} -\lambda & 0 & 1 \\ 0 & 2-\lambda & -1 \\ 3 & 0 & 5-\lambda \end{matrix}$$

- Examples: Find eigenpairs for
  1. A = [7/3 4/3; 2/3 5/3]   (from a class example)
  2. A = [2 1; 0 3]
  3. A = [-1 4; 2 -3]
* 4. A = [1 2; 3 4]   (has irrational evals)
* 5. A = [28 100; -9 -32]   (has repeated e-val with GM < AM)
  use the terms: eigenspace, basis for eigenspace
  view in app animating mapping v to Av
* 6. A = [-1 4 0; 2 -3 0; 1 0 2]
* 7. A = [1 3 0; 0 2 -1; 0 0 5]   diagonal
  Note: The matrix [0 0 1; 0 2 -1; 3 0 5] is not triangular
  8. A = [1 3; 3 1]   e-vals are -2, 4
  Note: here evecs form orthogonal basis of R^2
* 9. A = [1 4; -4 1]   e-vals are 1 \pm 4i
  Note: evals/evecs come in complex conjugate pairs

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 3 & 0 \\ 0 & 2-\lambda & -1 \\ 0 & 0 & 5-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)(5-\lambda)$$

Ex.) Find e-vals for $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2}$.

$$0 = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) - 6$$

$$= \lambda^2 - 5\lambda - 2$$ characteristic polynomial $= \det(A - \lambda I)$

By quadratic formula, zeros of char. poly. (eigenvals of A)

$$\lambda = \frac{5 \pm \sqrt{25 - 4(1)(-2)}}{2(1)} = \frac{5}{2} \pm \frac{\sqrt{33}}{2}.$$

Ex.) Find e-pairs of $A = \begin{bmatrix} 28 & 100 \\ -9 & -32 \end{bmatrix}$

$$0 = \det(A - \lambda I) = \begin{vmatrix} 28-\lambda & 100 \\ -9 & -32-\lambda \end{vmatrix}$$

$$= \lambda^2 + 4\lambda - 896 - (-900) = \lambda^2 + 4\lambda + 4$$

$$= (\lambda + 2)^2$$ (-2 is a double root)

Say $\lambda = -2$ is an e-value of algebraic multiplicity 2.

Now, to find e-vecs corresp. to $\lambda = -2$, find

$$\text{Null}(A - (-2)I) = \text{Null}\left(\begin{bmatrix} 30 & 100 \\ -9 & -30 \end{bmatrix}\right)$$

– i.e., solve $\begin{bmatrix} 30 & 100 \\ -9 & -30 \end{bmatrix} \vec{v} = \vec{0}$

$$\left[\begin{array}{cc|c} 30 & 100 & 0 \\ -9 & -30 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 10/3 & 0 \\ 0 & 0 & 0 \end{array}\right] \begin{array}{l} \to v_1 + \frac{10}{3} v_2 = 0 \\ \to 0 = 0 \end{array}$$

leave off?

v_2 free

e-vectors look like

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -\frac{10}{3} v_2 \\ v_2 \end{bmatrix} = \frac{v_2}{3} \begin{bmatrix} -10 \\ 3 \end{bmatrix}$$

v_2 free makes $\frac{v_2}{3}$ free · roaming in $\mathbb{R}$

E-vecs corresp. to $\lambda = -2$ are scalar mults. of $\begin{bmatrix} -10 \\ 3 \end{bmatrix}$.

So, the eigenspace corresp. to $\lambda = -2$ is a line (one-dimensional).

Each eval has both an AM (alg. mult., referred to earlier) and a geometric mult. GM = dimension of its eigenspace.

In this example, $\lambda = -2$ has AM $= 2$, GM $= 1$.

Whenever an e-val has AM > GM, that e-val is <u>degenerate</u>.

Ex.) Same instructions, now with

$$A = \begin{bmatrix} -1 & 4 & 0 \\ 2 & -3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

$$0 = \det\left(A - \lambda I\right) = \begin{vmatrix} -1-\lambda & 4 & 0 \\ 2 & -3-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix}$$

$$= 0 \cdot C_{13} + 0 \cdot C_{23} + (2-\lambda)(-1)^{3+3} \begin{vmatrix} -1-\lambda & 4 \\ 2 & -3-\lambda \end{vmatrix}$$

$$= (2-\lambda)\left[(-1-\lambda)(-3-\lambda) - 8\right] = (2-\lambda)(\lambda^2 + 4\lambda - 5)$$

$$= (2-\lambda)(\lambda+5)(\lambda-1) \implies 3 \text{ different e-vals}$$

$$\lambda = -5, 1, 2$$

E-vecs. corresp. to

$\underline{\lambda = -5}$: Solve $\begin{bmatrix} 4 & 4 & 0 \\ 2 & 2 & 0 \\ 1 & 0 & 7 \end{bmatrix} \vec{v} = \vec{0}$

$$\begin{bmatrix} 4 & 4 & 0 \\ 2 & 2 & 0 \\ 1 & 0 & 7 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -7 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \to v_1 = -7v_3 \\ \to v_2 = 7v_3 \\ \end{matrix}$$

v_3 free

e-vecs corresp. to $\lambda = -5$: $v_3 \begin{bmatrix} -7 \\ 7 \\ 1 \end{bmatrix}$ (so $\begin{bmatrix} -7 \\ 7 \\ 1 \end{bmatrix}$ is a basis for E_{-5} = Eigenspace for $\lambda = -5$)

Linear Independence of Functions

The material here is important both now (for one of the homework problems, Exercise 1.6.8, on the next hand-checked assignment), and later, when we are solving differential equations.

Review of linear independence

The definition of linear independence is this: a collection $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ of objects from the same "space" is **linearly independent** if the only linear combination

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k$$

which produces the *zero* object in the space is when all the weights $c_1, \ldots, c_k$ are zero.

To date the objects involved have been vectors, taken from $\mathbb{R}^n$ for some n. Examples include collections such as $S = \{(1,1,0),(1,0,1)\}$, which is linearly independent since the only weights c_1, c_2 which make the following equation true are $c_1 = c_2 = 0$:

$$c_1\begin{bmatrix}1\\1\\0\end{bmatrix} + c_2\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

However, the collection $S = \{(1,-2,2),(3,4,-1),(1,18,-12)\}$ is linearly dependent, because

$$\underline{\ \ 5\ \ }\begin{bmatrix}1\\-2\\2\end{bmatrix} + \underline{\ \ -2\ \ }\begin{bmatrix}3\\4\\-1\end{bmatrix} + \underline{\ \ 1\ \ }\begin{bmatrix}1\\18\\-12\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

Of course, the task of trying to fill in blanks with scalars until either

- you find a set of scalars, not all of which are zero, that works, or
- you decide the only set of scalars that works is the one with all zeros

is not the best way of deciding which of the scenarios, *linear independence* or *linear dependence* holds. Gaussian elimination on the augmented matrix $[\mathbf{A} \mid \mathbf{0}]$ tells you vectors $\mathbf{x}$ of weights which can be used to produce $\mathbf{0}$ from the columns of $\mathbf{A}$. And, if $\mathbf{A}$ is *square* and you only want to know if the columns are linearly independent or not, $\det(\mathbf{A})$ tells you all you need to know: $S = \{(1,-2,2),(3,4,-1),(1,18,-12)\}$ is linearly dependent since

$$\begin{vmatrix}1 & 3 & 1\\-2 & 4 & 18\\2 & -1 & -12\end{vmatrix} = 0.$$

Test for linear independence of k vectors from $\mathbb{R}^k$. Form the k-by-k (square) matrix $\mathbf{A}$ with the vectors to be tested as the columns of $\mathbf{A}$. If $|\mathbf{A}| \neq 0$, then the vectors are linearly independent; if $|\mathbf{A}| = 0$, then the vectors are linearly dependent.

Linear independence of functions

Functions are like vectors. The main difference is the number of coordinates. A vector from $\mathbb{R}^3$ has 3 coordinates. A function defined on the domain $a < x < b$ has a "coordinate" (a y-value) at each of the infinitely many x in the domain. The zero in this "vector space of functions defined on (a, b)" is the function $g(x) = 0$ which has the value 0 for each x.

When we have a collection of functions defined on a common domain, we can ask if that collection is linearly independent over that domain. It is, again, a question about weights in a linear combination to produce zero (the zero function). For example, $\{1, \sin^2 x, \cos^2 x\}$ is a linearly dependent collection on domain $(-\infty, \infty)$ since

$$\underline{\ \ -1\ \ }(1) + \underline{\ \ 1\ \ }(\sin^2 x) + \underline{\ \ 1\ \ }(\cos^2 x) \;=\; 0, \qquad \text{for every } -\infty < x < \infty.$$

That is, a collection of weights exists that produces the zero function without all the weights being zero. But, as before, hunting for scalars to weight the various functions can feel like looking for the proverbial needle in the haystack.

If you liked the simplicity of the process for testing independence of k vectors from $\mathbb{R}^k$ (see the box above), then here is an alternative for functions which mirrors it. It rests on the idea that, if the weights $(c_1, c_2, \ldots, c_k)$ in the linear combination of sufficiently-differentiable functions in the set $S = \{f_1, f_2, \ldots, f_k\}$ produce the zero function

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_k f_k(x) \;=\; 0, \qquad \text{for all } a < x < b,$$

then the same weights applied to linear combinations of derivatives, 2^{nd} derivatives, etc. also produce the zero function:

$$\begin{aligned}
c_1 f_1'(x) + c_2 f_2'(x) + \cdots + c_k f_k'(x) &= 0 \\
c_1 f_1''(x) + c_2 f_2''(x) + \cdots + c_k f_k''(x) &= 0 \\
&\vdots \\
c_1 f_1^{(j)}(x) + c_2 f_2^{(j)}(x) + \cdots + c_k f_k^{(j)}(x) &= 0 \\
&\vdots
\end{aligned}$$

If we include only those equations up to the $(k-1)^{\text{st}}$ derivative, we get a system of k equations in k unknowns, which we can arrange as the vector equation

$$c_1 \begin{bmatrix} f_1(x) \\ f_1'(x) \\ \vdots \\ f_1^{(k-1)}(x) \end{bmatrix} + c_2 \begin{bmatrix} f_2(x) \\ f_2'(x) \\ \vdots \\ f_2^{(k-1)}(x) \end{bmatrix} + \cdots + c_k \begin{bmatrix} f_k(x) \\ f_k'(x) \\ \vdots \\ f_k^{(k-1)}(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

or

$$\begin{bmatrix} f_1(x) & f_2(x) & \cdots & f_k(x) \\ f_1'(x) & f_2'(x) & \cdots & f_k'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(k-1)}(x) & f_2^{(k-1)}(x) & \cdots & f_k^{(k-1)}(x) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Test for linear independence of sufficiently-differentiable functions on domain (a, b). If the determinant

$$W(f_1, f_2, \ldots, f_k)(t) := \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_k(x) \\ f_1'(x) & f_2'(x) & \cdots & f_k'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(k-1)}(x) & f_2^{(k-1)}(x) & \cdots & f_k^{(k-1)}(x) \end{vmatrix}$$

is zero for all $a < x < b$, then the set of functions $S = \{f_1, f_2, \ldots, f_k\}$ is **linearly dependent** on (a, b). On the other hand, if there is *any* x in (a, b) at which this determinant $W(f_1, \ldots, f_k)(x) \neq 0$, then S is **linearly independent** on (a, b).

The determinant above is called the **Wronskian** of the functions $\{f_1, \ldots, f_k\}$.

Example 1:

Above we showed that $\{1, \sin^2 x, \cos^2 x\}$ is linearly *dependent* on $\mathbb{R}$. What about $\{1, \sin x, \cos x\}$? The Wronskian is

$$\begin{aligned} W(1, \sin, \cos)(x) &= \begin{vmatrix} 1 & \sin x & \cos x \\ 0 & \cos x & -\sin x \\ 0 & -\sin x & -\cos x \end{vmatrix} = \begin{vmatrix} \cos x & -\sin x \\ -\sin x & -\cos x \end{vmatrix} \\ &= -\cos^2 x - \sin^2 x = -1, \end{aligned}$$

for all real x. These functions are linearly independent on $(-\infty, \infty)$.

■

For more examples, follow this link https://youtu.be/zw9rkAD3BEI?t=181.