$$A = \begin{bmatrix} 28 & 100 \\ -9 & -32 \end{bmatrix}$$

$\lambda = -2$ repeated $(AM = 2)$
$GM = 1$ (degenerate)

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Math 231, Thu 25-Feb-2021 -- Thu 25-Feb-2021
Differential Equations and Linear Algebra
Spring 2020


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Thursday, February 25th 2021
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Wk 4, Th
Topic:: Wronskian and L.I. Functions
Topic:: DE classifications and models

       Look at "x maps to Ax" app

*   5. A = [28 100; -9 -32]    (has repeated e-val with GM < AM)
       view in app animating mapping v to Av
*   9. A = [1 4; -4 1]    e-vals are  1 \pm 4i
       Note: evals/evecs come in complex conjugate pairs
       Look at "x maps to Ax" app
*  10. A = [3 -2; 4 -1]    e-vals are  1 \pm 2i
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$$A = \begin{bmatrix} 1 & 4 \\ -4 & 1 \end{bmatrix}$$ Find e-pairs

Char. poly $= \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 4 \\ -4 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - (-16)$

So, set $(1-\lambda)^2 + 16 = 0 \Rightarrow (1-\lambda)^2 = -16$

$\Rightarrow 1 - \lambda = \pm\sqrt{-16} = \pm 4i$

$\Rightarrow \lambda = 1 \pm 4i$

Char. poly $2^{nd}$-degree — roots are complex conj. pairs.

Eigenvector corresp. to $\lambda = 1+4i$ — same as the null space of $A - (1+4i)I$

$$A - (1+4i)I = \begin{bmatrix} -4i & 4 \\ -4 & -4i \end{bmatrix} \rightsquigarrow \begin{bmatrix} -i & 1 \\ 0 & 0 \end{bmatrix}$$

$\uparrow$ $v_2$ free

$-iv_1 + v_2 = 0 \qquad v_1 = -iv_2 \qquad$ (or equivalently $v_2 = iv_1$)

e-vecs $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -iv_2 \\ v_2 \end{bmatrix} = v_2 \begin{bmatrix} -i \\ 1 \end{bmatrix}$

Now, for $1-4i$ (e-val., complex conj. of $1+4i$), a basis e-vec looks like the complex conj. of that for $1+4i$

For $1+4i$ basis e-vec $\begin{bmatrix} -i \\ 1 \end{bmatrix}$

For $1-4i$ basis e-vec. $\begin{bmatrix} i \\ 1 \end{bmatrix}$

Ex.) $A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \qquad 0 = \det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{vmatrix}$

$$= \lambda^2 - 2\lambda - 3 + 8 = \lambda^2 - 2\lambda + 5$$

$$\lambda = \frac{1}{2}\left(2 \pm \sqrt{4-20}\right) = \frac{1}{2}\left(2 \pm \sqrt{-16}\right) = 1 \pm 2i$$

For $1+2i$: Solve

$$[A - (1+2i)I]\vec{v} = \vec{0}$$

$$A - (1+2i)I = \begin{bmatrix} 3-(1+2i) & -2 \\ 4 & -1-(1+2i) \end{bmatrix}$$

$$= \begin{bmatrix} 2-2i & -2 \\ 4 & -2-2i \end{bmatrix} \leftrightarrow \begin{bmatrix} 2-2i & -2 \\ 0 & 0 \end{bmatrix} \to (2-2i)v_1 - 2v_2 = 0$$

$$2v_2 = (2-2i)v_1 \longrightarrow v_2 = (1-i)v_1$$

consider this one to be arb. chosen

E-vecs.

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ (1-i)v_1 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 1-i \end{bmatrix}$$

For $1-2i$, basis e-vector $\begin{bmatrix} 1 \\ 1+i \end{bmatrix}$

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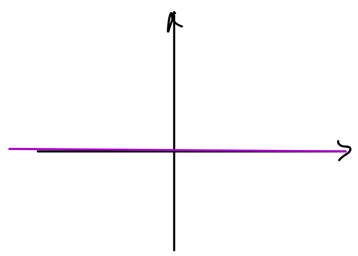
Linear ind. of functions

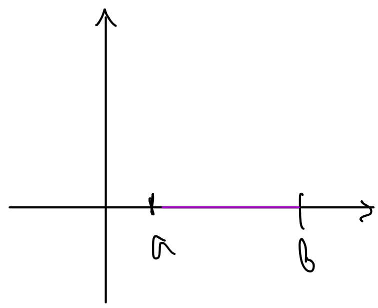
The fns. $\{1, x, x^2\}$ are L.I. as fns. on domain $(-\infty, \infty)$ because if you try to write the zero fn. as a linear combination

$$0 = \underbrace{\underline{0} \cdot 1 + \underline{0} \cdot x + \underline{0}\, x^2}_{\text{needs to be matched by this side's graph}}$$



But, $\{1, \sin^2 x, \cos^2 x\}$ are Linearly Dependent on $(a, b)$ $(a < b)$, since

$$0 = \underbrace{\underline{-1} \cdot 1 + \underline{1} \sin^2 x + \underline{1} \cos^2 x}_{\text{can reproduce 0-graph w/ not-all-zero weights.}}$$



# Linear Independence of Functions

The material here is important both now (for one of the homework problems, Exercise 1.6.8, on the next hand-checked assignment), and later, when we are solving differential equations.

## Review of linear independence

The definition of linear independence is this: a collection $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ of objects from the same "space" is **linearly independent** if the only linear combination

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k$$

which produces the *zero* object in the space is when all the weights $c_1, \ldots, c_k$ are zero.

To date the objects involved have been vectors, taken from $\mathbb{R}^n$ for some $n$. Examples include collections such as $S = \{(1,1,0), (1,0,1)\}$, which is linearly independent since the only weights $c_1, c_2$ which make the following equation true are $c_1 = c_2 = 0$:

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

However, the collection $S = \{(1,-2,2), (3,4,-1), (1,18,-12)\}$ is linearly dependent, because

$$\underline{\quad 5 \quad} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} + \underline{\quad -2 \quad} \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix} + \underline{\quad 1 \quad} \begin{bmatrix} 1 \\ 18 \\ -12 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Of course, the task of trying to fill in blanks with scalars until either

- you find a set of scalars, not all of which are zero, that works, or
- you decide the only set of scalars that works is the one with all zeros

is not the best way of deciding which of the scenarios, *linear independence* or *linear dependence* holds. Gaussian elimination on the augmented matrix $[\mathbf{A} \mid \mathbf{0}]$ tells you vectors $\mathbf{x}$ of weights which can be used to produce $\mathbf{0}$ from the columns of $\mathbf{A}$. And, if $\mathbf{A}$ is *square* and you only want to know if the columns are linearly independent or not, $\det(\mathbf{A})$ tells you all you need to know: $S = \{(1,-2,2), (3,4,-1), (1,18,-12)\}$ is linearly dependent since

$$\begin{vmatrix} 1 & 3 & 1 \\ -2 & 4 & 18 \\ 2 & -1 & -12 \end{vmatrix} = 0.$$

or

$$\begin{bmatrix} f_1(x) & f_2(x) & \cdots & f_k(x) \\ f_1'(x) & f_2'(x) & \cdots & f_k'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(k-1)}(x) & f_2^{(k-1)}(x) & \cdots & f_k^{(k-1)}(x) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

**Test for linear independence of sufficiently-differentiable functions on domain** $(a, b)$. If the determinant

$$W(f_1, f_2, \ldots, f_k)(t) := \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_k(x) \\ f_1'(x) & f_2'(x) & \cdots & f_k'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(k-1)}(x) & f_2^{(k-1)}(x) & \cdots & f_k^{(k-1)}(x) \end{vmatrix}$$

is zero for all $a < x < b$, then the set of functions $S = \{f_1, f_2, \ldots, f_k\}$ is **linearly dependent** on $(a, b)$. On the other hand, if there is *any* $x$ in $(a, b)$ at which this determinant $W(f_1, \ldots, f_k)(x) \neq 0$, then $S$ is **linearly independent** on $(a, b)$.

The determinant above is called the **Wronskian** of the functions $\{f_1, \ldots, f_k\}$.

### Example 1:

Above we showed that $\{1, \sin^2 x, \cos^2 x\}$ is linearly *dependent* on $\mathbb{R}$. What about $\{1, \sin x, \cos x\}$? The Wronskian is

$$\begin{aligned} W(1, \sin, \cos)(x) &= \begin{vmatrix} 1 & \sin x & \cos x \\ 0 & \cos x & -\sin x \\ 0 & -\sin x & -\cos x \end{vmatrix} = \begin{vmatrix} \cos x & -\sin x \\ -\sin x & -\cos x \end{vmatrix} \\ &= -\cos^2 x - \sin^2 x = -1, \end{aligned}$$

for all real $x$. These functions are linearly independent on $(-\infty, \infty)$.

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For more examples, follow this link https://youtu.be/zw9rkAD3BEI?t=181.

**Test for linear independence of $k$ vectors from $\mathbb{R}^k$.** Form the $k$-by-$k$ (square) matrix $\mathbf{A}$ with the vectors to be tested as the columns of $\mathbf{A}$. If $|\mathbf{A}| \neq 0$, then the vectors are linearly independent; if $|\mathbf{A}| = 0$, then the vectors are linearly dependent.

## Linear independence of functions

Functions are like vectors. The main difference is the number of coordinates. A vector from $\mathbb{R}^3$ has 3 coordinates. A function defined on the domain $a < x < b$ has a "coordinate" (a $y$-value) at each of the infinitely many $x$ in the domain. The zero in this "vector space of functions defined on $(a, b)$" is the function $g(x) = 0$ which has the value 0 for each $x$.

When we have a collection of functions defined on a common domain, we can ask if that collection is linearly independent over that domain. It is, again, a question about weights in a linear combination to produce zero (the zero function). For example, $\{1, \sin^2 x, \cos^2 x\}$ is a linearly dependent collection on domain $(-\infty, \infty)$ since

$$\underline{\quad -1 \quad}(1) + \underline{\quad 1 \quad}(\sin^2 x) + \underline{\quad 1 \quad}(\cos^2 x) \;=\; 0, \qquad \text{for every } -\infty < x < \infty.$$

That is, a collection of weights exists that produces the zero function without all the weights being zero. But, as before, hunting for scalars to weight the various functions can feel like looking for the proverbial needle in the haystack.

If you liked the simplicity of the process for testing independence of $k$ vectors from $\mathbb{R}^k$ (see the box above), then here is an alternative for functions which mirrors it. It rests on the idea that, if the weights $(c_1, c_2, \ldots, c_k)$ in the linear combination of sufficiently-differentiable functions in the set $S = \{f_1, f_2, \ldots, f_k\}$ produce the zero function

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_k f_k(x) \;=\; 0, \qquad \text{for all } a < x < b,$$

then the same weights applied to linear combinations of derivatives, 2nd derivatives, etc. also produce the zero function:

$$\begin{array}{rcl}
c_1 f_1'(x) + c_2 f_2'(x) + \cdots + c_k f_k'(x) & = & 0 \\
c_1 f_1''(x) + c_2 f_2''(x) + \cdots + c_k f_k''(x) & = & 0 \\
 & \vdots & \\
c_1 f_1^{(j)}(x) + c_2 f_2^{(j)}(x) + \cdots + c_k f_k^{(j)}(x) & = & 0 \\
 & \vdots &
\end{array}$$

If we include only those equations up to the $(k-1)^{\text{st}}$ derivative, we get a system of $k$ equations in $k$ unknowns, which we can arrange as the vector equation

$$c_1 \begin{bmatrix} f_1(x) \\ f_1'(x) \\ \vdots \\ f_1^{(k-1)}(x) \end{bmatrix} + c_2 \begin{bmatrix} f_2(x) \\ f_2'(x) \\ \vdots \\ f_2^{(k-1)}(x) \end{bmatrix} + \cdots + c_k \begin{bmatrix} f_k(x) \\ f_k'(x) \\ \vdots \\ f_k^{(k-1)}(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$