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Math 231, Mon 8-Mar-2021 -- Mon 8-Mar-2021
Differential Equations and Linear Algebra
Spring 2020


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Monday, March 8th 2021
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Wk 6, Mo
Topic:: DE models
Topic:: 1st-order DEs
HW:: WW
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More 1st-Order DE Models

Turning word problems into DE models

1. Determine what is changing, perhaps through a keyword like *derivative*, *rate*, *growth*, *decay*, *increase*, etc.
2. Consider whether there are any principles or physical laws governing the dynamics:

   Newton's laws,
   net change = (rate of input) - (rate of outgo),
   etc.
3. Write a differential equation, one that is an instantaneous statement which holds at any time.
4. Check that each of the terms have the same units.
5. Account for bits of information concerning what happens in the system at a specific time. These may be initial conditions, or may help in the determination of various model constants.

Example 1: Falling object, Example 2, p. 12, Boyce & DiPrima, 9th Ed.

Assume the drag force on a falling body is proportional to its speed. Taking downward direction to be positive, get

$$m\frac{dv}{dt} = mg - \gamma v.$$

Suppose the object has mass 10 kg and drag coefficient $\gamma = 2$ kg/s. If dropped from a height of 300 m, how fast will the object be moving at impact?

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Example 2: Trench, Exercise 18, p.139

A person deposits \$25,000 in a bank that pays 5% per year interest, compounded continuously. The person withdraws \$15 per week from the account. Write a DE model for $V(t)$, the amount in the account at time t.

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Example 3: Newton's Law of Cooling

It seems reasonable that, in certain situations, the rate at which the temperature $T(t)$ of an object changes should be proportional to the difference between body temperature T and ambient temperature A. (Here we are assuming that, at all times, whatever heat the body has is instantaneously uniformly distributed throughout the body, so that there are no internal spatial temperature differences. We also assume that heat loss to the surroundings do not change the surrounding temperature.) This is confirmed by experiment. Write a differential equation in the body's temperature T. If a thermometer reading 60°F is taken outside, and readings of 70°F and 76°F are recorded 10 and 20 minutes later, respectively, what is the outside temperature?

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Example 4: Mixing Model

There is a pool at Herculane (a spa in SW Romania) used for rheumatic treatment. When half-filled, it contains 50,000 gallons of natural hot springwater with salt mixed in. Suppose this half-full tank currently has 5000 lbs. of Herculane salt. Fresh water is being allowed into the pool at the rate of 2000 gal/hr. Let us assume it mixes instantly with the hot springwater (so that the concentration of salt is spatially uniform). If, at the same time, briny water is allowed to leave the tank at a rate of 1000 gal/hr, what is the salt concentration at the instant the tank becomes completely filled?

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Example 5: Weight Gain[1]

[1]**Differential Equation Models, Vol. 1**, Braun, Martin, Courtney S. Coleman, Donald A. Drew, Eds., Springer-Verlag, 1978, pp. 9–10.

A man eats a diet of 2500 cal/day; 1200 of them go to basal metabolism (i.e., get used up automatically). He spends approximately 16 cal/kg/day times his body weight (in kg) in weight-proportional exercise. Assume that the storage of calories as fat is 100% efficient and that 1 kg fat contains 10,000 cal. Find how his weight varies with time. What is his weight's **steady state**?

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Solving 1st-order linear DEs

Is linear: $p(t)\,y' + q(t)\,y = r(t)$

Here: t ind var.

y dep. var.

y, y'. no y'' or higher deriv $\longrightarrow$ 1st order

y, y' both present simply to 1st-power w/ coeffs not dependent on y or y' $\longrightarrow$ linear DE

Dividing by $p(t)$:

$$y' = \frac{-q(t)}{p(t)}\,y + \frac{r(t)}{p(t)}$$ or, after relabeling

$$(\ast) \qquad y' = a(t)\,y + f(t)$$

Every 1st-order linear ODE may be put in this form

Further, if $f(t) = 0$, would further classify this form ($\star$) as homogeneous.

Special case: homogeneous version ($\star$)

$$y' = a(t)\, y$$

Note: This is separable, so we divide and separate

$$\frac{1}{y}\frac{dy}{dt} = a(t)$$

$$\frac{1}{y}\, dy = a(t)\, dt$$

Now integrate

$$\int \frac{1}{y}\, dy = \int a(t)\, dt \implies \ln|y| = C + \int a(t)\, dt$$

Making both sides exponent for e

$$e^{\ln|y|} = e^{C + \int a(t)\, dt}$$

$$|y(t)| = e^{C} \cdot e^{\int a(t)\, dt}$$

$$\boxed{y(t) = \tilde{C} e^{\int a(t)\, dt},} \qquad \tilde{C} \text{ any real no.}$$

solves my DE, infinite family of solns. w/ single degree freedom

Think of this like a 1-dimensional "space" of fns with basis function

$$\varphi(t) = e^{\int a(t)\,dt}$$

Nonhomogeneous version:

$$y' = a(t)\, y + f(t)$$

Guess form of a solution

Have $\varphi(t) = e^{\int a(t)\,dt}$ solves the DE when $f(t) = 0$.

So, let me propose this guess

$$y_p(t) = u(t)\varphi(t) \quad \text{w/ } u(t) \text{ a fn. to be determined.}$$

Criterion for determining $u(t)$: Demand $y_p(t)$ solves the problem

$$\boxed{y'} = a(t) \boxed{y} + f(t).$$

Proposed y_p: has deriv. $\frac{d}{dt} y_p = \frac{d}{dt}\left[u(t)\,\varphi(t)\right]$

$$= u'\varphi + u\varphi'$$

Insert into my DE both y_p, y_p'

$$u'\varphi + u\varphi' = a\left[u\varphi\right] + f$$

$$u'\varphi + u\left[\varphi' - a\varphi\right] = f$$

Here, note that $\varphi' = a\varphi$ by design. We found φ so as to solve $y' = ay$. So, get simplified version

$$u'\varphi = f \implies u' = \frac{f}{\varphi}$$

$$\implies u(t) = \int \frac{f(t)}{\varphi(t)}\,dt$$

So,
$$y_p = u(t)\cdot\varphi(t) = \varphi(t)\cdot\int \frac{f(t)}{\varphi(t)}\,dt$$

Final result:

$y' = a(t)\,y + f(t)$ has soln (family)

$$y(t) = \varphi\cdot\left[\int \frac{f(t)}{\varphi(t)}\,dt + C\right]$$

$$= \underbrace{C\,\varphi(t)}_{\text{solns. to homog. problem}} + \underbrace{\varphi(t)\int \frac{f(t)}{\varphi(t)}\,dt}_{\text{called } y_p}$$

Similar?

When, in Ch. 1, solving $Ax = b$ and problem was

- consistent
- A a free col.

the result was

$$\underbrace{\begin{bmatrix} \\ \\ \\ \end{bmatrix}}_{\text{particular vector}} + b\,\underbrace{\begin{bmatrix} \\ \\ \\ \end{bmatrix}}_{\text{basis vector in Null}(A)}$$

Ex.) Newton's Law of Cooling

$$\boxed{\frac{dT}{dt} = -k(T - E)}$$

k, E constants

Set $y = T$

$$y' = -ky + kE$$

so linear, nonhomogeneous

$a(t) = -k$ (constant)

$f(t) = kE$ (constant)

$$\varphi(t) = e^{\int a(t)\,dt} = e^{-kt}$$

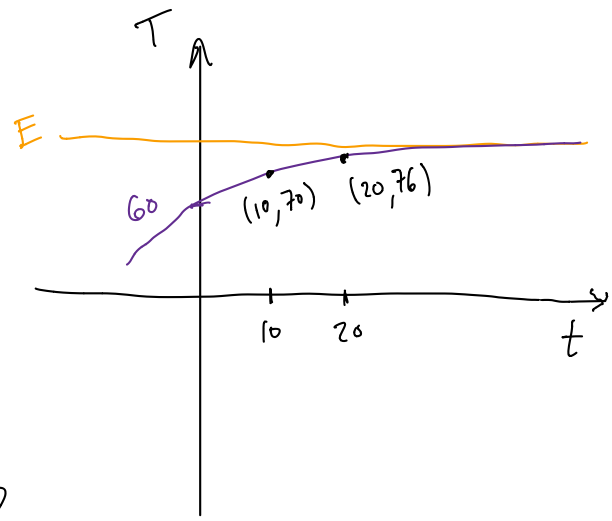
$$y_p(t) = \varphi(t) \cdot \int \frac{f(t)}{\varphi(t)}\,dt = e^{-kt} \cdot \int \frac{kE}{e^{-kt}}\,dt$$

$$= e^{-kt} \cdot kE \int e^{kt}\,dt = e^{-kt} \cdot E \cdot e^{kt} = E$$

So, our solution is

$$T(t) = y(t) = C \cdot \varphi(t) + y_p(t)$$

$$\boxed{T(t) = Ce^{-kt} + E}$$



$T(0) = 60$

$T(10) = 70$

$T(20) = 76$

Use to get C, k, E ?