Math 231, Wed 10-Mar-2021 -- Wed 10-Mar-2021
Differential Equations and Linear Algebra
Spring 2020

Wednesday, March 10th 2021

Topic:: 1st order linear ODEs wrapup
Topic:: Existence and uniqueness

mixing problem

A "chemical-in-tank" (salt) model from last class notes:
 There is a pool at Heculane (in Romania) used for rheumatic treatment.
 When half-filled, it contains 50,000 gallons of natural hot springwater
 with salt mixed in. Suppose this half-full tank currently has 5000 lbs
 of Herculane salt. Fresh water is being allowed into the pool at the
 rate of 2000 gal/hr. Let us assume it mixes instantly with the hot
 springwater (so that the concentration of salt is spatially uniform).
 If, at the same time, briny water is allowed to leave the tank at a
 rate of 1000 gal/hr, what is the salt concentration at the instant the
 tank becomes completely filled?

in lbs

 Let x(t) be amount of salt in tank at time t. Then

$\frac{5000}{50000}$ at start

DE model x' = (rate of influx of salt) - (rate of outflux of salt)

dx/dt

units lbs/hr.

$$= \quad 0 \quad - \left(1000\ \frac{\text{gal}}{\text{hr}}\right)\left(\frac{x}{50000+1000t}\ \text{lbs}/\text{gal}\right)$$

Note: Those notes state another problem about about caloric intake,
 energy use, and weight gain. This is really another tank problem.

 Let x(t) = man's mass in kg at time t

 10000 x' = 2500 - 1200 - 16x = 1300 - 16x

Nonlinear DEs?

$$\boxed{\frac{dx}{dt} = -\frac{1000x}{50000+1000t}, \qquad x(0) = 5000}$$

$$\frac{dx}{dt} = \frac{-x}{50+t} = a(t)\,x + f(t)$$ to be linear

$$a(t) = \frac{-1}{50+t} \qquad f(t) = 0$$ so homogeneous

Homogeneous linear

Last time had full-on (perhaps nonhomogeneous) 1st-order linear ODE

$$y' = a(t)\, y + f(t)$$

had solution

$$\underset{\substack{\uparrow \\ \text{homog.} \\ \text{soln.}}}{y_h} + \underset{\substack{\uparrow \\ \text{particular} \\ \text{soln.}}}{y_p}$$

can be ignored if $f(t) = 0$

w/ $$y_h(t) = C\, \underbrace{e^{\int a(t)\,dt}}_{\varphi(t)}, \qquad y_p(t) = \varphi(t) \int \frac{f(t)}{\varphi(t)}\,dt$$

$$\underline{\varphi(t)} = e^{\int a(t)\,dt} = e^{\int \frac{-1}{50+t}\,dt} = e^{-\int \frac{dt}{50+t}}$$

$$= e^{-\ln|50+t|} = e^{\ln\left(|50+t|^{-1}\right)} = |50+t|^{-1}$$

So, our solution

$$x(t) = \frac{C}{|50+t|}$$

By $5000 = x(0) = \dfrac{C}{|50+0|} \Longrightarrow C = 250000$

So, $x(t) = 250000/(50+t)$ soln. to $\underline{\text{IVP}}$

IC = initial condition (above: $x(0) = 5000$)

DE = differential eqn. (above: $\frac{dx}{dt} = \frac{-x}{50+t}$)

IVP = initial value problem

= DE + IC

1. How do we know if a DE has a soln?

$$y' = g(t, y)$$

2. Furthermore, if it does, is it unique? – Certainly not!

Ex.) $y' = 2x \Rightarrow y = x^2 + C$

Ex.) $y'' = 7 - 2e^x \Rightarrow y' = 7x - 2e^x + C_1$

$$\Rightarrow y = \frac{7}{2}x^2 - 2e^x + C_1 x + C_2$$

Existence and Uniqueness

Uniqueness of solutions for ODEs is not plausable.

But, if we turn our problem into an IVP

$$y' = f(t, y), \quad \text{subject to IC} \quad y(t_0) = y_0. \tag{1}$$

we might have more satisfactory answers to these questions:

1. Does problem (1) have a solution? (Existence)
2. Does problem (1) have *at most one* solution? (Uniqueness)
3. On what interval does our solution solve the problem?

To partially address these questions, we have the following theorems.

Theorem 1 (Existence): Suppose $f(t, y)$ is continuous in an open rectangle $R : a < t < b, c < y < d$ of the ty-plane. Given any point (t_0, y_0) of R, there exists a solution of problem (1) on some open interval I containing t_0.

Theorem 2 (Uniqueness): If, in addition to the assumptions of Theorem 1, the partial derivative $\partial f / \partial y$ is continuous throughout R, then the solution of (1) is unique.

While these two theorems answer the most important fundamental questions—those of existence and uniqueness of a solution—they are silent on the interval of existence for that solution.

Partial deriv.

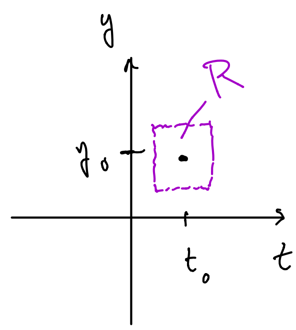
- applicable when more than 1 ind. var.
- achieved by treating other vars. like constants

$$y(t, x) = x^2 t - \cos(2x)$$

$$\frac{\partial y}{\partial x} = 2xt + 2\sin(2x)$$

treat other ind. var, t, like a constant

$$\frac{\partial y}{\partial t} = x^2$$



Thm. 1 says: If "solving" $\begin{cases} y' = g(t,y) \\ y(t_0) = y_0 \end{cases}$

and we can find a rectangle R

- that encloses the point (t_0, y_0), and
- inside which $f(t, y)$ is continuous,

then a solution exists.

Thm. 2: If, in addition, $\frac{\partial f}{\partial y}$ is continuous inside R, then there is only one solution.

There is, however, a stronger theorem that applies to the linear problem

$$y' = a(t)y + f(t), \qquad y(t_0) = y_0, \tag{2}$$

and addresses all three questions.

> **Theorem 3:** If the functions a, f are continuous on an open interval I containing the number t_0, then there exists a unique function $\tilde{y}$ that satisfies both parts (the DE and the IC for arbitrary y_0) of (2). Moreover, the interval of existence (i.e., the t values for which the DE is satisfied by $\tilde{y}$) includes all of I.

Q: Is integration a technique that works in general to solve (1) whenever a solution exists?

Q: Consider the given differential equation, along with initial condition $y(x_0) = y_0$. Identify the set of points (x_0, y_0), or indicate that none exist, for which the Fundamental Existence/Uniqueness Theorem for 1st order IVPs *does not guarantee* a unique solution passes through them.

(a) $y' = \dfrac{e^x + y}{x^2 + y^2}$

(b) $y' = 2xy + \sqrt{x}$

(c) $y' = \dfrac{2x + 3y}{x - 4y}$

(d) $y' = \dfrac{\cos y}{x - 1}$