There is, however, a stronger theorem that applies to the linear problem

$$y' = a(t)y + f(t), \qquad y(t_0) = y_0, \tag{2}$$

and addresses all three questions.

> **Theorem 3:** If the functions a, f are continuous on an open interval I containing the number t_0, then there exists a unique function $\tilde{y}$ that satisfies both parts (the DE and the IC for arbitrary y_0) of (2). Moreover, the interval of existence (i.e., the t values for which the DE is satisfied by $\tilde{y}$) includes all of I.

Q: Is integration a technique that works in general to solve (1) whenever a solution exists?

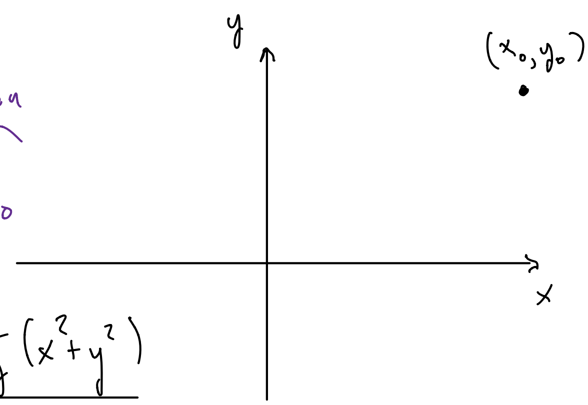
Q: Consider the given differential equation, along with initial condition $y(x_0) = y_0$. Identify the set of points (x_0, y_0), or indicate that none exist, for which the Fundamental Existence/Uniqueness Theorem for 1st order IVPs *does not guarantee* a unique solution passes through them.

(a) $y' = \dfrac{e^x + y}{x^2 + y^2}$

(b) $y' = 2xy + \sqrt{x}$

(c) $y' = \dfrac{2x + 3y}{x - 4y}$

(d) $y' = \dfrac{\cos y}{x - 1}$



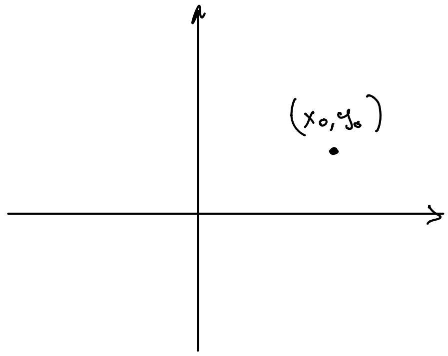
(a) $y' = g(x,y) = \boxed{\dfrac{e^x + y}{x^2 + y^2}}$, $\overbrace{y(x_0) = y_0}^{\text{avoid origin}}$

$$\frac{\partial g}{\partial y} = \frac{\frac{\partial}{\partial y}(e^x + y)\cdot(x^2+y^2) - (e^x+y)\frac{\partial}{\partial y}(x^2+y^2)}{(x^2+y^2)^2}$$

$$= \boxed{\frac{1\cdot(x^2+y^2) - (e^x+y)\cdot 2y}{(x^2+y^2)^2}}$$

Q: Where is $g(x,y)$ discontinuous?
" " $\frac{\partial g}{\partial y}$ discontinuous? } For both, at $(0,0)$

(b) $y' = 2xy + \sqrt{x} = g(x,y)$



$$\frac{\partial g}{\partial y} = 2x$$

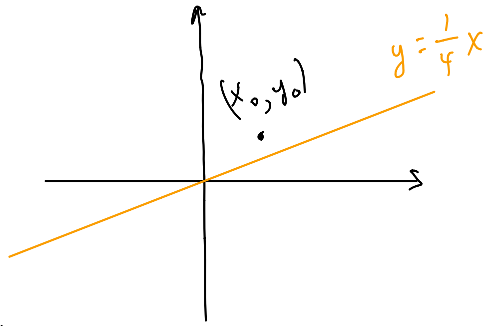
Q: Where are g, $\frac{\partial g}{\partial y}$ discont.

bad when $x < 0$

cont. everywhere

Have existence & uniquess of solns. if we add an IC $y(x_0) = y_0$ w/ $x_0 > 0$

(c) $y' = \dfrac{2x + 3y}{x - 4y}$, $y(x_0) = y_0$



$$g(x,y) = \frac{2x + 3y}{x - 4y}$$

$$\frac{\partial g}{\partial y} = \frac{3(x-4y) + 4(2x+3y)}{(x-4y)^2}$$

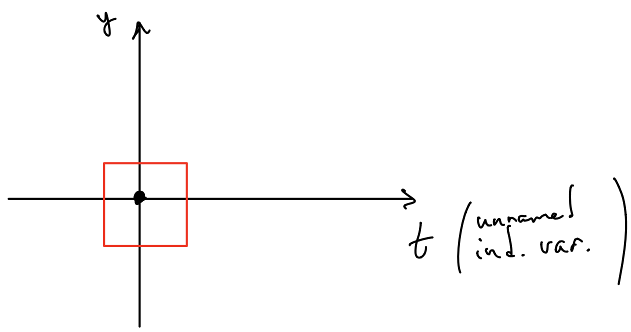
where is

1. g disc?
2. $\partial g/\partial y$ disc?

Ans. to both: Along the line $y = \frac{1}{4}x$

Problems exist where

- there isn't any soln.
- there are multiple solns



Ex.) $y' = 2\sqrt{y}$, $y(0) = 0$

$g(t,y) = 2\sqrt{y}$, $\quad \frac{\partial g}{\partial y} = \frac{1}{\sqrt{y}}$

bad when $y < 0$

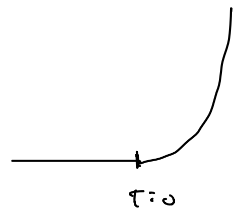
bad when $y \leq 0$

Note:

- The theorems don't apply
- The IVP has more than one solution, Here's two

$y_1(t) = 0$ for all t

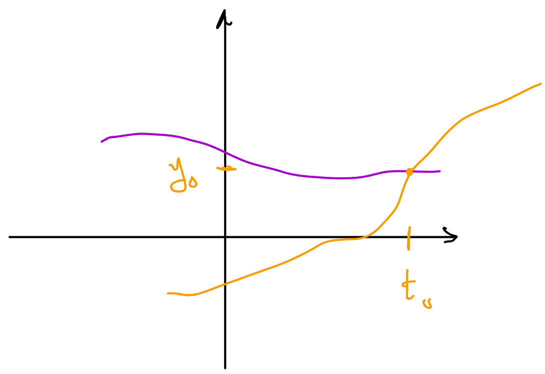
$$y_2(t) = \begin{cases} 0 & \text{when } t < 0 \\ t^2, & \text{when } t \geq 0 \end{cases}$$



Note: If two solutions to

$y' = f(t,y)$

happen to cross somewhere



Ex.) $y' = 2y + 5t^2 - 3t + 2$ Solve it.

1^{st}-order linear $a(t) = 2$

$f(t) = 5t^2 - 3t + 2$

Our method:

1. Find $\varphi(t) = e^{\int 2\,dt} = e^{2t}$
2. Find $\boxed{y_p = \varphi(t) \cdot \int \frac{f(t)}{\varphi(t)}\,dt}$ Variation of Parameters Formula

Here, this formula gives

$$e^{2t} \cdot \int \frac{5t^2-3t+2}{e^{2t}}\, dt$$

Need to calculate

$$\int (5t^2 - 3t + 2)\, e^{-2t}\, dt$$

An alternate way to find y_p:

a) Guess a form for it w/ constants to-be-determined

b) Solve for constants

Have problem $\quad y' = 2y + 5t^2 - 3t + 2$, or

$$y' - 2y = 5t^2 - 3t + 2$$

Guess:

$$y_p(t) = At^2 + Bt + C \qquad \text{(proposal)}$$

$$y_p' = 2At + B$$

Insert into my DE

$$\underbrace{(2At + B)}_{y'} - 2\underbrace{(At^2 + Bt + C)}_{y} = \underbrace{5t^2 - 3t + 2}_{\text{target}}$$

$$-2At^2 + (2A - 2B)t + (B - 2C) = 5t^2 - 3t + 2$$

t^2 $\quad t^1$ $\quad t^0$

Now equate coeffs for "like" term

	LHS		RHS
t^2	$-2A$	$=$	5
t^1	$2A-2B$	$=$	-3
t^0	$B-2C$	$=$	2

use to find A, B, C

Put together as before

Soln.

$$\underbrace{C \cdot e^{2t}}_{C \cdot \phi(t)} + \underbrace{At^2 + Bt + C}_{y_p \text{ found}}$$