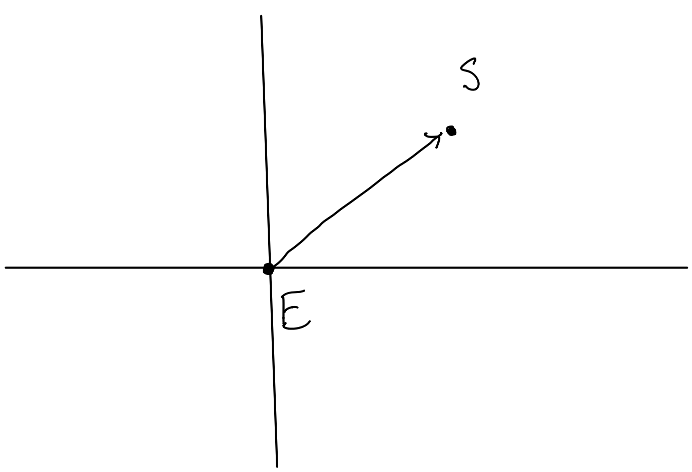
Example 6: Numerical solution of Volterra Predator–Prey Eqns

Suppose $a = 1$, $b = 0.5$, $c = -0.75$ and $d = 0.25$ in the **predator–prey** equations (1). Use 4^{th} order RK to find the solution passing through the IC $x(0) = 1$, $y(0) = 2$.

Note that, while the lines of code for the algorithm above must be implemented, they do little until more code is introduced to initiate the specifics of the problem at hand and to call the vrk routine. In the case of our problem, we run commands such as these:

```
var('t x y')
f(t,x,y) = (x - x*y/2, -0.75*y + 0.25*x*y)
keepCoords = [1,2]
pts = vrk(f, [1,2], 0, 7, 200, keepCoords)
list_plot(pts, plotjoined=True)    # plots the soln (x(t),y(t)) in phase plane
```

■



$$\begin{bmatrix} m\dfrac{d^2x}{dt^2} \\ m\dfrac{d^2y}{dt^2} \end{bmatrix} = G \begin{bmatrix} \dfrac{mM}{x^2+y^2} \\ \dfrac{mM}{x^2+y^2} \end{bmatrix}$$

error

Earth: mass M, always at $(0,0)$

Satellite: at position $(x(t), y(t)) = (\text{dist.})(\text{unit vector})$

$$= \sqrt{x^2+y^2} \cdot \frac{\langle x, y\rangle}{\sqrt{x^2+y^2}}$$

magnitude direction

proportional to

$$m\vec{a} \propto \frac{mM}{(\text{dist})^2}$$

Higher order systems

The study of 1^{st} order systems of DEs is far more important than it may appear at first glance. The reason is that both higher order scalar DEs and higher order systems of DEs can always be recast as 1^{st} order systems. Section 7.1 in the text discusses this conversion process at some length. We demonstrate the process here with a single example.

Example 7: The Two-Body Problem

Assume a "planet" (or other heavenly body) of mass M is fixed at point $(0,0)$. There is a satellite of mass m orbiting this planet, whose position we label $(x,y) = (x(t), y(t))$. The gravitational force between planet and satellite as felt by the satellite has

$$\text{magnitude} = \frac{GMm}{x^2+y^2}, \qquad \text{and direction vector} \qquad \text{direction} = \frac{(-x,-y)}{(x^2+y^2)^{1/2}}.$$

By Newton's Law $F = ma$, we obtain the system of 2^{nd} order DEs

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= \frac{-GMx}{(x^2+y^2)^{3/2}}, \\ \frac{d^2y}{dt^2} &= \frac{-GMy}{(x^2+y^2)^{3/2}}. \end{aligned}\right\} \text{ system, } 2^{\text{nd}}\text{-order} \qquad (2)$$

To convert this to a 1^{st} order system, introduce new dependent variables $u = dx/dt$, $v = dy/dt$. The symbols d^2x/dt^2 and d^2y/dt^2 in system (2) may now be replaced by du/dt and dv/dt respectively, yielding this 1^{st} order system:

Now 4 dep. vars. u, v, x, y

$$\begin{bmatrix} dx/dt \\ du/dt \\ dy/dt \\ dv/dt \end{bmatrix} = \begin{bmatrix} u \\ -GMx/(x^2+y^2)^{3/2} \\ v \\ -GMy/(x^2+y^2)^{3/2} \end{bmatrix}$$

$$\begin{aligned} \frac{dx}{dt} &= u, \quad \text{invented name} \\ \frac{du}{dt} &= \frac{-GMx}{(x^2+y^2)^{3/2}}, \\ \frac{dy}{dt} &= v, \quad \text{invented name} \\ \frac{dv}{dt} &= \frac{-GMy}{(x^2+y^2)^{3/2}}. \end{aligned} \qquad (3)$$

To solve this problem numerically using RK4, and graph the motion of the satellite in time, we carry out the following commands, in which we assume $G = M = 1$, as well as these ICs:

$$x(0) = 1, \quad u(0) = 0, \quad y(0) = 0, \quad v(0) = 0.75.$$

```
var('t x y')
f(t,x,dx,y,dy) = (dx, -x/(x^2+y^2)^(3/2), dy, -y/(x^2+y^2)^(3/2))
keepCoords = [1,3]
pts = vrk(f, [1,0,0,0.75], 0, 3, 100, keepCoords)
list_plot(pts, plotjoined=True)
```

A 1^{st} order system in n dep. vars.

$$\vec{x} = \langle x_1, x_2, \dots, x_n \rangle,$$

when writeable in normal form looks like

$$\frac{d\vec{x}}{dt} = \vec{f}(t, \vec{x}) \quad \text{or} \quad \begin{bmatrix} dx_1/dt \\ dx_2/dt \\ \vdots \\ dx_n/dt \end{bmatrix} = \begin{bmatrix} f_1(t, \vec{x}) \\ f_2(t, \vec{x}) \\ \vdots \\ f_n(t, \vec{x}) \end{bmatrix}$$

same

Sometimes,

$\vec{f}(t, x)$ is expressible as $A(t)\vec{x} + \vec{f}(t)$ } linear 1^{st}-order systems

Ex.)

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} ax_1 - bx_1x_2 \\ -cx_2 + dx_1x_2 \end{bmatrix}$$

predator-prey (nonlinear)

<u>cannot</u> be written as

$$\begin{bmatrix} \_ & \_ \\ \_ & \_ \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \_ \\ \_ \end{bmatrix}$$

where my blanks are fns. of t only.

But

Ex.)
$$\underbrace{\begin{bmatrix} x_1' \\ x_2' \end{bmatrix}}_{d\vec{x}/dt} = \begin{bmatrix} 3x_1 + 2t\,x_2 - \cos t \\ 2x_1 - x_2 + t^2 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 3 \\ 2 \end{bmatrix} x_1 + \begin{bmatrix} 2t \\ -1 \end{bmatrix} x_2} + \begin{bmatrix} -\cos t \\ t^2 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 3 & 2t \\ 2 & -1 \end{bmatrix}}_{A(t)} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} + \underbrace{\begin{bmatrix} -\cos t \\ t^2 \end{bmatrix}}_{\vec{f}(t)}$$

So this is a linear system

Note: ① When $\vec{f}(t)$ = zero vector, say our linear system is homogeneous.

② when $A(t) = A$ (constant matrix), say we have constant-coefficient system

Going forward, we'll be solving $\frac{d\vec{x}}{dt} = A\vec{x}$

constant   no $\vec{f}(t)$

Constant A no $f(t)$

Recall from Chapter 2 (when only one dep. var)

$$x' = a(t)\,x + f(t)$$

- When $f(t) = 0$ called this DE homogeneous
- $a(t)$ is 1-1 matrix
- Soln. to $x' = a(t)\,x$ is

$$\varphi = e^{\int a(t)\,dt} \quad \text{(or any scalar mult. of it)}$$

$\Longrightarrow$ In the scalar case $x' = a(t)\,x$, solns are exponentials.

Now $\dfrac{d\vec{x}}{dt} = A_{n\times n}\,\vec{x}$ (still homog.)

might guess solns are exponential:

$$\vec{x}(t) = e^{\lambda t}\,\vec{v}$$