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Math 231, Wed 17-Mar-2021 -- Wed 17-Mar-2021
Differential Equations and Linear Algebra
Spring 2020

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Wednesday, March 17th 2021
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Wk 7, We
Topic:: Existence and uniqueness wrapup
Read:: ODELA 3.2-3.4
HW::   HC03 due Mar. 23

Problem x' = Ax
 - homogeneous, constant coefficient version of
   x' = A(t)x + f(t)
 - how it looks written out as a system of equations
 - why eigenpairs of A are relevant

Solve
 1. x' = [1 1; 3 -1] x,    x(0) = [4; 0]
 2. x' = [2 1; 1  2] x
 3. x' = [0 2 4; -5 -11 -20; 2 4 7] x

Show direction fields
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Problem: n dep. vars. $x_1, \ldots, x_n$, 1^{st}-order, w/ 1^{st}-deriv. solved for

most general 1^{st}-order system of linear DEs in n dep. vars.

non-dep. term

$$\begin{cases} x_1' = a_{11}(t)x_1 + \cdots + a_{1n}(t)x_n + f_1(t) \\ x_2' = a_{21}(t)x_1 + \cdots + a_{2n}(t)x_n + f_2(t) \\ \vdots \\ x_n' = a_{n1}(t)x_1 + \cdots + a_{nn}(t)x_n + f_n(t) \end{cases}$$

$$\begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & \cdots & a_{2n}(t) \\ \vdots & & \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$

First Order Linear, Homogeneous Systems with Constant Coefficients

The problems we are solving here are, for some positive integer $n > 0$, of the form

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \qquad \text{or, more simply} \qquad \mathbf{x}' = \mathbf{A}\mathbf{x}. \tag{1}$$

We have seen that, for solutions of the form $e^{\lambda t}\mathbf{v}$ to exist (where $\mathbf{v}$ is a vector in $\mathbb{R}^n$), it is necessary that $(\lambda, \mathbf{v})$ be an eigenpair of $\mathbf{A}$. If we can find n linearly independent solutions of this form

$$e^{\lambda_1 t}\mathbf{v}_1, \quad e^{\lambda_2 t}\mathbf{v}_2, \quad \ldots, \quad e^{\lambda_n t}\mathbf{v}_n,$$

then these solutions form a **fundamental set of solutions** to (1), giving us its general solution

$$\mathbf{x}(t) \;=\; c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 + \cdots + c_n e^{\lambda_n t}\mathbf{v}_n.$$

Here are some useful facts to know.

Theorem 1: Suppose $\mathbf{A}$ is an n-by-n matrix with entries that are real numbers.

1. If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $\mathbf{A}$, then $\det(\mathbf{A}) = \lambda_1\lambda_2\cdots\lambda_n$, the product of eigenvalues. [This part of the theorem is, in fact, true even in the more general case where entries of $\mathbf{A}$ are complex numbers.]

2. If $\lambda = \alpha + i\beta$ (with α, β both real and $\beta \neq 0$) is an eigenvalue of $\mathbf{A}$ with corresponding eigenvector $\mathbf{v} = \mathbf{u} + i\mathbf{w}$ (where all the entries of $\mathbf{u}$ and $\mathbf{w}$ are real numbers), then the complex conjugate $\overline{\lambda} = \alpha - i\beta$ is an eigenvalue of $\mathbf{A}$ as well, with corresponding eigenvector $\mathbf{u} - i\mathbf{w}$.

3. If the eigenvalues $\lambda_1, \ldots, \lambda_n$ are n *distinct* complex numbers, with corresponding eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$, then $\{e^{\lambda_1 t}\mathbf{v}_1, e^{\lambda_2 t}\mathbf{v}_2, \ldots, e^{\lambda_n t}\mathbf{v}_n\}$ is a fundamental set of solutions to $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

4. If, for each eigenvalue of $\mathbf{A}$, the geometric multiplicity equals the algebraic multiplicity, then by choosing a basis of eigenvectors corresponding to each eigenvalue and amassing them into the collection $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, one again obtains a fundamental set of solutions $\{e^{\lambda_1 t}\mathbf{v}_1, e^{\lambda_2 t}\mathbf{v}_2, \ldots, e^{\lambda_n t}\mathbf{v}_n\}$. (Here, λ_j is the eigenvalue that goes with eigenvector $\mathbf{v}_j$.)

5. If $\mathbf{A}$ is a **symmetric** matrix (that is, $a_{ij} = a_{ji}$ for each $1 \leqslant i, j \leqslant n$), then the eigenvalues are all *real* numbers whose geometric multiplicities equal their algebraic multiplicities. Moreover, eigenvectors corresponding to *distinct* eigenvalues are orthogonal, and there exists an *orthogonal* basis of $\mathbb{R}^n$ consisting of eigenvectors of $\mathbf{A}$.

Most of the matrices $\mathbf{A}$ whose eigenpairs we have calculated have fallen into case 3 of this theorem, giving us, in theory, a fundamental set of of solutions to $\mathbf{x}' = \mathbf{A}\mathbf{x}$. The one true exception was the matrix

$$\mathbf{A} \;=\; \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

for which one eigenvalue, (-1) had algebraic multiplicity two but geometric multiplicity one. It is in cases such as these that we must work hardest to obtain a fundamental set of solutions.

Direction fields

For an *autonomous* 1st order (perhaps nonlinear) system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where $\mathbf{A}$ is n-by-n, with $n = 2$ or $n = 3$, it possible to draw a **direction field** in the appropriate **phase space** (called the **phase plane** when $n = 2$). The idea in the $n = 2$ linear case is that, at any point $\mathbf{x} = (x_1, x_2)$, we have

$$\begin{pmatrix} dx_1/dt \\ dx_2/dt \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad \Rightarrow \qquad \frac{dx_2}{dx_1} = \frac{dx_2/dt}{dx_1/dt} = \frac{a_{21}x_1 + a_{22}x_2}{a_{11}x_1 + a_{12}x_2}.$$

(The same idea works, with only slight modification, in the case of nonlinear autonomous 1st order systems.) One can place a hash mark with slope dx_2/dx_1 at the point (x_1, x_2). It is, of course, convenient to hand this task over to a software package. See the `PPLANE` applet at `http://math.rice.edu/%7edfield/dfpp.html`.

Example 1:

Look at direction fields for

1. the nonlinear system

$$\begin{aligned} dx/dt &= 2x - y + 3(x^2 - y^2) + 2xy, \\ dy/dt &= x - 3y - 3(x^2 - y^2) + 3xy, \end{aligned} \qquad \text{or} \qquad \begin{aligned} dx/dt &= 1 - x - \frac{4xy}{1 + x^2}, \\ dy/dt &= x\left(1 - \frac{y}{1 + x^2}\right). \end{aligned}$$

   The former is the default when the `PPLANE` applet starts up. The latter was introduced in a paper by Lengyel & Epstein from 1991 related to their study of the chlorine dioxide-iodine-malonic acid (ClO_2-I_2-MA) reaction.

2. $\dfrac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$
3. $\dfrac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$

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Classifying equilibrium solutions for $\mathbf{x}' = \mathbf{A}\mathbf{x}$

Sticking to the linear case, let us assume, for the moment, that $\det(\mathbf{A}) \neq 0$. We have an equilibrium point $\mathbf{x}$ when the rates of change $dx_1/dt, \ldots dx_n/dt$ are simultaneously zero—that is, whenever $\mathbf{x}' = \mathbf{A}\mathbf{x} = \mathbf{0}$. When $\det(\mathbf{A}) = 0$ there are infinitely many equilibrium points, but as we are assuming $\det(\mathbf{A}) \neq 0$, $\mathbf{x} = \mathbf{0}$ is the only one. We wish, now, to classify this equilibrium point. Our discussion will focus on systems in which the matrix **A** is 2-by-2, but the ideas extend to higher dimensions. We will look at two important cases now, and return to cover other cases later.

Example 2: **A** has real eigenvalues of opposite sign

$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$. Plot the direction field and use the general solution to explain what it shows. The origin is classified as a **saddle point**.

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Example 3: **A** has real, distinct eigenvalues of same (positive) sign

$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$. Plot the direction field and use the general solution to explain what it shows. The origin is classified as an **unstable node**.

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Example 4: **A** has real, distinct eigenvalues of same (negative) sign

$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -5 & 2 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$. Plot the direction field and use the general solution to explain what it shows. The origin is classified as an **asymptotically stable node**.

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Guess Monday that perhaps sols. of

$$\begin{bmatrix} x_1' \\ \vdots \\ x_n' \end{bmatrix} = A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

might be exponential form:
$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} = e^{\lambda t} \underset{\substack{\uparrow \\ \text{const} \\ \text{vector}}}{\vec{v}}$$

If such a guess works, then

$$\text{LHS: } \frac{d}{dt}\vec{x}(t) \overset{\text{our guess}}{=} \frac{d}{dt}\left(e^{\lambda t}\vec{v}\right) = \lambda e^{\lambda t} \cdot \vec{v}$$
(constant-mult. rule of differentiation)

$$\text{RHS: } A\vec{x}(t) \overset{\text{insert guess}}{=} A \cdot \left(e^{\lambda t}\vec{v}\right) = e^{\lambda t} A\vec{v}$$

So, for our guess to solve $\vec{x}' = A\vec{x}$, we require

$$\lambda e^{\lambda t}\vec{v} = e^{\lambda t} A\vec{v}$$

or, dividing by $e^{\lambda t}$, get

$$\lambda \vec{v} = A\vec{v}$$

So this yields a nontrivial result precisely when $(\lambda, \vec{v})$ form an eigenpair of A.

Ex. $\quad \vec{x}' = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \vec{x}$, subj. to $\vec{x}(0) = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$

(Note: Same as problem

$$x_1' = x_1 + x_2, \qquad x_1(0) = 4$$

$$x_2' = 3x_1 - x_2, \qquad x_2(0) = 0 \quad)$$

First, find e-vals of $A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$. Solve

$$0 = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda) - 3$$

$$= \lambda^2 - 1 - 3 = \lambda^2 - 4 = (\lambda - 2)(\lambda + 2)$$

$\Rightarrow$ e-vals $\lambda = -2, 2$.

Corresp. to $\lambda = -2$: $\text{Null}(A - (-2)I)$ contains e-vectors

$$\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} \vec{v} = \vec{0}$$

free col.

$$\left[\begin{array}{cc|c} 3 & 1 & 0 \\ 3 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 3 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow 3v_1 + v_2 = 0$$

$$v_1 = -\frac{1}{3} v_2$$

e-vectors corresp. to $\lambda = -2$ look like

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} v_2 \\ v_2 \end{bmatrix} = \frac{1}{3} v_2 \underbrace{\begin{bmatrix} -1 \\ 3 \end{bmatrix}}$$

representative (basis) for e-vectors in E_{-2}.

For $\lambda = 2$

$$(A - 2I) = \begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \to -v_1 + v_2 = 0$$

Corresp. e-vectors

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_2 \\ v_2 \end{bmatrix} = v_2 \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

basis vector for E_2.

Obtained e-pairs:

λ	basis e-vector
-2	$\begin{bmatrix} -1 \\ 3 \end{bmatrix}$
2	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

and our earlier derivation says

$$e^{-2t}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -e^{-2t} \\ 3e^{-2t} \end{bmatrix} \quad \text{solves} \quad \vec{x}\,' = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}\vec{x}$$

and so does

$$e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix}$$

However, when $t = 0$ is plugged in to either one

$$t = 0: \quad \begin{bmatrix} -e^{-2(0)} \\ 3e^{-2(0)} \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \quad \underline{\text{Not}} \quad \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$t=0: \quad \begin{bmatrix} e^{2(0)} \\ e^{2(0)} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \underline{\text{Not}} \quad \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

So neither soln on its own satisfies the IC.

What does (while still solving $\vec{x}' = A\vec{x}$) is an appropriately-chosen linear combination

$$\vec{x}(t) = \underbrace{c_1 e^{-2t} \begin{bmatrix} -1 \\ 3 \end{bmatrix} + c_2}_{\text{must choose } c_1, c_2} e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Choose using the IC: Need

$$\begin{bmatrix} 4 \\ 0 \end{bmatrix} = \vec{x}(0) = c_1 \begin{bmatrix} -1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Use GE to get c_1, c_2:

$$\left[\begin{array}{cc|c} -1 & 1 & 4 \\ 3 & 1 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 3 \end{array}\right]$$

So $c_1 = -1$, $c_2 = 3$

and the soln. is

$$\vec{x}(t) = -1\, e^{-2t}\begin{bmatrix}-1\\3\end{bmatrix} + 3\, e^{2t}\begin{bmatrix}1\\1\end{bmatrix}$$

$$= \begin{bmatrix} e^{-2t} + 3e^{2t} \\ -3e^{-2t} + 3e^{2t}\end{bmatrix} \quad \checkmark \qquad \vec{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix}$$

Ex.

$$\vec{x}\,' = \begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\vec{x} \iff \left(\begin{aligned} x_1' &= 2x_1 + x_2 \\ x_2' &= x_1 + 2x_2\end{aligned}\right)$$

Find e-pairs:

λ	basis e-vectors
3	$\begin{bmatrix}1\\1\end{bmatrix}$
1	$\begin{bmatrix}1\\-1\end{bmatrix}$

General soln. is any linear comb. of

$$e^{3t}\begin{bmatrix}1\\1\end{bmatrix}, \quad e^{t}\begin{bmatrix}1\\-1\end{bmatrix}$$

so

$$c_1 e^{3t}\begin{bmatrix}1\\1\end{bmatrix} + c_2 e^{t}\begin{bmatrix}1\\-1\end{bmatrix}.$$

(as far as we can go w/out IC)