Math 231, Thu 18-Mar-2021 -- Thu 18-Mar-2021
Differential Equations and Linear Algebra
Spring 2020

Thursday, March 18th 2021

Wk 7, Th
Topic:: Fund'l matrix and Wronskian
Read:: ODELA 3.5

Wronskian $= \det\left(\Phi(t)\right)$

Fundamental set of solutions

In both Chapters 2 and 3, we encounter the homogeneous linear problem

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}(t)\mathbf{x},$$

$\boxed{\frac{dy}{dt} = a(t)\,y}$ ~~$+f(t)$~~ zero when homog.

where $\mathbf{x}(t) = \langle x_1(t), x_2(t), \ldots, x_n(t)\rangle$ is a vector function, meaning that for each input t, $\mathbf{x}(t)$ is in $\mathbb{R}^n$.

Ch. 2 →
- Case: $n = 1$:
  This is the 1-dimensional case studied in Chapter 2, where the "matrix" $\mathbf{A}(t)$ is 1-by-1 whose only entry is $a(t)$. The solution of (1) is

  $$x(t) = \varphi(t)c,$$

  $\varphi(t) = e^{\int a(t)\,dt}$

  where $c \in \mathbb{R}$ is arbitrary, representing one degree of freedom.

- Case: $n > 1$, $\mathbf{A}(t) = \mathbf{A}$ (a constant n-by-n matrix):
  Examples so far include

  Vector problem (1st-order, linear, homog., constant coeff)

  $\vec{x}\,' = A\vec{x}$

  - $\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \mathbf{x}$

    The eigenvalue $\lambda = -2$ has eigenspace E_{-2} with basis vector $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

    The eigenvalue $\lambda = 2$ has eigenspace E_2 with basis vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

    The general solution:

    $\vec{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

    $$\mathbf{x}(t) = c_1 e^{-2t}\begin{bmatrix} -1 \\ 3 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -e^{-2t} & e^{2t} \\ 3e^{-2t} & e^{2t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

    $\Phi(t)$ = fundamental matrix solution (Its columns solve the original 1st-order system)

  - $\mathbf{x}' = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{x}$

    The eigenvalue $\lambda = 1$ has eigenspace E_1 with basis vector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

    The eigenvalue $\lambda = 3$ has eigenspace E_3 with basis vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

    The general solution:

$$\vec{x}_h(t) = c_1 e^{t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -e^{t} & e^{3t} \\ e^{t} & e^{3t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \Phi(t)\cdot\vec{c}$$

Note: $\Phi(0) = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$

has columns from basis e-vectors

- $\mathbf{x}' = \begin{bmatrix} 0 & 2 & 4 \\ -5 & -11 & -20 \\ 2 & 4 & 7 \end{bmatrix} \mathbf{x}$

The eigenvalue $\lambda = -2$ has eigenspace E_{-2} with basis vector $\begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix}$

The eigenvalue $\lambda = -1$ has eigenspace E_{-1} with basis vectors $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$

~~The general solution:~~

To see this

$$\det\left(\begin{bmatrix} 0 & 2 & 4 \\ -5 & -11 & -20 \\ 2 & 4 & 7 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right) = \begin{vmatrix} -\lambda & 2 & 4 \\ -5 & -11-\lambda & -20 \\ 2 & 4 & 7-\lambda \end{vmatrix}$$

$$= (-\lambda)(-1)^2 \begin{vmatrix} -11-\lambda & -20 \\ 4 & 7-\lambda \end{vmatrix} + 2(-1)^3 \begin{vmatrix} -5 & -20 \\ 2 & 7-\lambda \end{vmatrix} + 4(-1)^4 \begin{vmatrix} -5 & -11-\lambda \\ 2 & 4 \end{vmatrix}$$

$$= -\lambda\left[(-11-\lambda)(7-\lambda) - (-80)\right] - 2\left[(-5)(7-\lambda) - (-40)\right] + 4\left[-20 - 2(-11-\lambda)\right]$$

$= \text{Cubic poly.} = \text{some alg. (factoring?)} = (\lambda+2)(\lambda+1)^2 \cdot \text{constant}$

$E_{-1} = \text{Null}(A+I)$

$$\overset{A+I}{\begin{bmatrix} 1 & 2 & 4 \\ -5 & -10 & -20 \\ 2 & 4 & 8 \end{bmatrix}} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2 free cols.

$x_1 + 2x_2 + 4x_3 = 0$ w/ 2 freedoms

basis which has 2 L.I. eigenvectors

So

$$e^{-2t}\begin{bmatrix}1\\-5\\2\end{bmatrix}, \quad e^{-t}\begin{bmatrix}-2\\1\\0\end{bmatrix}, \quad e^{-t}\begin{bmatrix}-4\\0\\1\end{bmatrix}$$

all solve $\vec{x}\,' = A\vec{x}$ and so does every linear combination

$$c_1 e^{-2t}\begin{bmatrix}1\\-5\\2\end{bmatrix} + c_2 e^{-t}\begin{bmatrix}-2\\1\\0\end{bmatrix} + c_3 e^{-t}\begin{bmatrix}-4\\0\\1\end{bmatrix}$$

$$\vec{x}_h(t) = \underbrace{\begin{bmatrix} e^{-2t} & -2e^{-t} & -4e^{-t} \\ -5e^{-2t} & -e^{-t} & 0 \\ 2e^{-2t} & 0 & e^{-t} \end{bmatrix}}_{\Phi(t)} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

general solution

General solutions

We say $\mathbf{x}_h(t) = \mathbf{\Phi}(t)\mathbf{c}$ is the **general solution** for $\mathbf{x}' = \mathbf{A}\mathbf{x}$ on an open interval $I = (a, b)$ if all solutions of the latter take the form of the former—i.e., if all solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ are writeable as linear combinations of the columns of $\mathbf{\Phi}(t)$. This is true for the examples above, and is true whenever all these criteria are met:

- $\mathbf{\Phi}(t)$, like $\mathbf{A}$, is n-by-n square.
- each column of $\mathbf{\Phi}(t)$ is a solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$.
- the matrix $\mathbf{\Phi}(t)$ is nonsingular for $t \in I$. This is needed so that, for any $t_0 \in I$, a unique choice of vector $\mathbf{c} = \langle c_1, c_2, \ldots, c_n \rangle$ exists so that the initial condition $\mathbf{x}(t_0) = \mathbf{k}$ can be met, regardless of $\mathbf{k} \in \mathbb{R}^n$.

In presence of this IC, need to solve

$$\vec{k} = \Phi(t_0)\vec{c}$$

for $\vec{c}$.

Some deep insights:

1. Eigenvectors corresponding to distinct eigenvalues are linearly independent.
2. In our constructions above, $\mathbf{\Phi}(0)$ is simply a matrix whose columns are basis eigenvectors from all the eigenspaces. So long as no eigenvalue is degenerate, the last fact means that $\mathbf{\Phi}(0)$ is nonsingular.
3. **Abel's Theorem**: If the columns of $\mathbf{\Phi}(t)$ all solve $\mathbf{x} = \mathbf{A}\mathbf{x}$ on the open interval $I = (a, b)$, then either
    - $\mathbf{\Phi}(t)$ is singular at every $t \in I$, or
    - $\mathbf{\Phi}(t)$ is nonsingular at every $t \in I$. In particular, if $0 \in I$ and $\mathbf{\Phi}(0)$ is nonsingular, that is enough to conclude $\mathbf{\Phi}(t)$ stays nonsingular throughout I.
4. For the constant-coefficient case, where $\mathbf{A}(t)$ is a constant matrix, the interval $I = (-\infty, \infty)$.

The upshot: So long as no eigenvalue of the n-by-n matrix $\mathbf{A}$ is degenerate, our construction leads to a general solution.

An adjustment to the method for nonreal eigenvalues

Euler's Formula

$$e^{it}$$

In Calculus

Maclaurin series

$$\begin{cases} e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\ \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \end{cases}$$

Use to get Euler's Formula (next time)