Math 231, Fri 19-Mar-2021 -- Fri 19-Mar-2021
Differential Equations and Linear Algebra
Spring 2020

Friday, March 19th 2021

Wk 7, Fr
Topic:: Nonreal eigenvalues
Topic:: Phase portraits
Read:: ODELA 3.5

Phase portraits

- axes for dependent vars only, not independent (parametrization)

$\vec{x}' = Ax$

$$\begin{bmatrix} dx_1/dt \\ \vdots \\ dx_n/dt \end{bmatrix} = A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

- equilibrium points, 2-dim'l systems
    includes all vectors x in null(A)
    origin is always one, and the only when A is nonsingular
    classifying equilibrium at origin

If $\vec{x} \in \text{Null}(A)$ you have an equilibrium pt.

A. stability
    globally asymptotically stable: all solutions have origin as limit
    unstable: ICs arbitrarily close to origin go infinitely far away
    stable: solns starting finite distance from origin remain finite dist.

B. type
    node: eigenvalues are all real and of the same sign
    saddle: eigenvalues are all real with some positive, some negative
    spiral: eigenvalues are nonreal
    center: special case of spiral, where eigenvalues are purely imaginary

- examples
    1. x' = [1 1; 3 -1] x
       gen'l soln: c1 e^{-2t}<1, -3> + c2 e^{2t}<1, 1>

$\lambda = -2, 2$

    2. x' = [2 1; 1 2] x
       gen'l soln: c1 e^{t}<-1, 1> + c2 e^{3t}<1, 1>

$\lambda = 1, 3$

unstable node

Euler's formula
When A has eigenvalue alpha + i beta and corresp. eigenvector u + iw

From previous days

$$x' = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \vec{x}, \quad \text{gen'l soln} \quad \vec{x} = c_1 e^{-2t} \begin{bmatrix} 1 \\ -3 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

as separate eqns,

$$x_1' = x_1 + x_2$$

$$x_2' = 3x_1 - x_2$$

Origin is a saddle point (<u>type</u>)

Occurs every time the two eigenvalues are

- real
- opposite sign.

When both e-values are

- real
- same sign

Call the origin a <u>node</u>.

$$e^{it} = 1 + (it) + \frac{1}{2!}(it)^2 + \frac{1}{3!}(it)^3 + \cdots$$

pure imag. exponent $i = \sqrt{-1}$, $t \in \mathbb{R}$

by Maclaurin series

$$= 1 + it + \frac{1}{2!}(-t^2) + \frac{1}{3!}(-it^3) + \cdots$$

even powers don't have i

$$= 1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \frac{1}{6!}t^6 + \cdots$$ — Maclaurin series for $\cos t$

$$+ \, i\left(t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \frac{1}{7!}t^7 + \cdots\right)$$

for $\sin t$

$$= \cos t + i \sin t.$$

Euler's Formula

$$e^{it} = \cos t + i \sin t$$

when applied to βt (instead of t), then $(-\beta t)$

$$\boxed{\begin{aligned} e^{i\beta t} &= \cos(\beta t) + i \sin(\beta t) \\ e^{i(-\beta t)} &= \cos(\beta t) - i \sin(\beta t) \end{aligned}}$$

Then

$$e^{(\alpha + i\beta)t} = e^{\alpha t} \cdot e^{i\beta t} = e^{\alpha t}\left[\cos(\beta t) + i \sin(\beta t)\right]$$

$$e^{(\alpha - i\beta)t} = e^{\alpha t} \cdot e^{-i\beta t} = e^{\alpha t}\left[\cos(\beta t) - i \sin(\beta t)\right]$$

In course of finding e-pairs of A, may get nonreal $\lambda, \vec{v}$:

$$\begin{cases} \lambda_1 = \alpha + i\beta, & \vec{v}_1 = \vec{u} + i\vec{w} \\ \lambda_2 = \alpha - i\beta, & \vec{v}_2 = \vec{u} - i\vec{w} \end{cases}$$

Goal: To use these eigenpairs in a general soln. but get rid of references to $i = \sqrt{-1}$.

Say produced these eigenpairs.

Were slated be columns in $\Phi(t)$

$$\left.\begin{array}{ll} ① & e^{(\alpha+i\beta)t}(\vec{u}+i\vec{w}) \\ ② & e^{(\alpha-i\beta)t}(\vec{u}-i\vec{w}) \end{array}\right\} \text{ both solve } \vec{x}' = A\vec{x}.$$

$$e^{(\alpha+i\beta)t}(\vec{u}+i\vec{w}) = e^{\alpha t}\cdot e^{i\beta t}(\vec{u}+i\vec{w})$$

$$= e^{\alpha t}\left[\cos(\beta t) + i\sin(\beta t)\right](\vec{u}+i\vec{w})$$

$$= e^{\alpha t}\left\{\left[\cos(\beta t)\vec{u} - \sin(\beta t)\vec{w}\right] + i\left[\sin(\beta t)\vec{u} + \cos(\beta t)\vec{w}\right]\right\}$$

Similar

$$e^{(\alpha-i\beta)t}(\vec{u}-i\vec{w}) = e^{\alpha t}\left\{\left[\cos(\beta t)\vec{u} - \sin(\beta t)\vec{w}\right] - i\left[\sin(\beta t)\vec{u} + \cos(\beta t)\vec{w}\right]\right\}$$

Instead, replace ① and ② by

$$\left.\begin{array}{ll} (\text{their sum})/2 & = e^{\alpha t}\left[\cos(\beta t)\vec{u} - \sin(\beta t)\vec{w}\right] \\ (\text{their diff})/(2i) & = e^{\alpha t}\left[\sin(\beta t)\vec{u} + \cos(\beta t)\vec{w}\right] \end{array}\right\} \text{ instead are used in } \Phi(t).$$

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3.  x' = [-1 -3; 6 5] x

4.  x' = [-15 -20 -20; 14 19 20; -6 -10 11] x
     has eigenpair  (-1), <0, 1, -1>
     has eigenpair  (-3-4i), <1, -1, 2/5 + i/5>
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Ex.

$$\vec{x}' = \begin{bmatrix} -1 & -3 \\ 6 & 5 \end{bmatrix} \vec{x}$$

$$\lambda_1 = 2 - 3i \quad \text{goes w/} \quad \begin{bmatrix} 1 \\ -1+i \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Take $\alpha = 2$, $\beta = -3$, $\vec{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Formulas for 2 independent solns (serve as columns in $\Phi(t)$)

$$e^{2t}\left\{\cos(-3t)\begin{bmatrix} 1 \\ -1 \end{bmatrix} - \sin(-3t)\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\} = e^{2t}\begin{bmatrix} \cos(3t) \\ \sin(3t) - \cos(3t) \end{bmatrix}$$

$$e^{2t}\left\{\sin(-3t)\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \cos(-3t)\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\} = e^{2t}\begin{bmatrix} -\sin(3t) \\ \sin(3t) + \cos(3t) \end{bmatrix}$$

using $\sin(-3t) = -\sin(3t)$
$\cos(-3t) = \cos(3t)$