Math 231, Mon 22-Mar-2021 -- Mon 22-Mar-2021
Differential Equations and Linear Algebra
Spring 2021

Monday, March 22nd 2021

Topic:: Homogeneous linear systems
Topic:: Degenerate eigenvalues

Warmup:

1. x' = [-1 -3; 6 5] x

$$\vec{x}' = \begin{bmatrix} -1 & -3 \\ 6 & 5 \end{bmatrix} \vec{x}$$

Did on Friday

eigenpairs: 2-3i with <1,-1> + <0,1> i
            2+3i with <1,-1> - <0,1> i

general solution:

x(t) = c_1 e^{2t}(cos(3t) <1,-1> + sin(3t) <0,1>)
     + c_2 e~~^{2t}(cos(3t) <1,-1> + sin(3t) <0,1>~~)

$$\vec{x}(t) = c_1 e^{2t}\left(\cos(3t)\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \sin(3t)\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + c_2 e^{2t}\left(\sin(3t)\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \cos(3t)\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

phase portrait (origin is unstable, of type "spiral point")

Note:

If trajectories were orbits, call origin a "center", stable
This happens precisely when eigenvalues of 2-by-2 A are purely imaginary.

$\alpha \pm i\beta$ with $\alpha = 0$

2. Suppose x' = Ax had
   eigenvalue 2, with basis eigenvectors <3, 1, 0, -1>, <2, 0, -1, 1>
   eigenvalue -1-i, with basis eigenvector <1+i, 2-3i, -4i, 7>
   Write the general solution

$\alpha = -1$
$\beta = -1$

Degenerate eigenvalues

Solutions corresp. to these statements

(1) $e^{2t}\begin{bmatrix} 3 \\ 1 \\ 0 \\ -1 \end{bmatrix}$, (2) $e^{2t}\begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$, $\cancel{e^{(-1-i)t}\begin{bmatrix} 1+i \\ 2-3i \\ -4i \\ 7 \end{bmatrix}}$, $\cancel{e^{(-1+i)t}\begin{bmatrix} 1-i \\ 2+3i \\ 4i \\ 7 \end{bmatrix}}$

replace with

$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 7 \end{bmatrix}$ $\vec{w} = \begin{bmatrix} 1 \\ -3 \\ -4 \\ 0 \end{bmatrix}$

(3) $e^{-t}\left(\cos(-t)\begin{bmatrix} 1 \\ 2 \\ 0 \\ 7 \end{bmatrix} - \sin(-t)\begin{bmatrix} 1 \\ -3 \\ -4 \\ 0 \end{bmatrix}\right)$, (4) $e^{-t}\left(\sin(-t)\begin{bmatrix} 1 \\ 2 \\ 0 \\ 7 \end{bmatrix} + \cos(-t)\begin{bmatrix} 1 \\ -3 \\ -4 \\ 0 \end{bmatrix}\right)$

(1) - (4) are columns for fund'l matrix $\Phi(t)$.

1st Order Linear Homogeneous Systems with Constant Coeffs $\mathbf{x}' = \mathbf{A}\mathbf{x}$

The case of nonreal eigenvalues

As we have seen, matrices with real-number entries can have nonreal eigenvalues. Suppose $\lambda = \alpha + i\beta$ is an eigenvalue with α, β real and $\beta \neq 0$; denote its complex conjugate by $\overline{\lambda} = \alpha - i\beta$. Let $\mathbf{v} = \mathbf{u} + i\mathbf{w}$ be an eigenvector corresponding to λ, with the entries of both $\mathbf{u}, \mathbf{w}$ being real numbers. We have seen that, using this notation, $(\lambda, \mathbf{v}) = (\alpha + i\beta, \mathbf{u} + i\mathbf{w})$ and $(\overline{\lambda}, \overline{\mathbf{v}}) = (\alpha - i\beta, \mathbf{u} - i\mathbf{w})$ are eigenpairs of $\mathbf{A}$.

Under the procedures we have learned for solving $\mathbf{x}' = \mathbf{A}\mathbf{x}$, we might include

$$\mathbf{x}_1(t) := e^{\lambda t}\mathbf{v} \qquad \text{and} \qquad \mathbf{x}_2(t) := e^{\overline{\lambda} t}\overline{\mathbf{v}}$$

in a fundamental set of solutions to $\mathbf{x}' = \mathbf{A}\mathbf{x}$. However, it might be preferable to replace these two solutions with two others that have no references to $i = \sqrt{-1}$. The way we do this is nearly identical to how we removed such references to i in the case of 2$^{\text{nd}}$ order linear homogeneous DEs. We have

$\alpha + \beta i$ w/ corresp. e-vector $\vec{u} + i\vec{w}$

$$\begin{aligned}
\mathbf{x}_1(t) &= e^{\lambda t}\mathbf{v} = e^{(\alpha+i\beta)t}(\mathbf{u} + i\mathbf{w}) = e^{\alpha t}[\cos(\beta t) + i\sin(\beta t)](\mathbf{u} + i\mathbf{w}) \\
&= e^{\alpha t}[\cos(\beta t)\mathbf{u} - \sin(\beta t)\mathbf{w}] + i\,e^{\alpha t}[\sin(\beta t)\mathbf{u} + \cos(\beta t)\mathbf{w}], \qquad \text{and} \\
\mathbf{x}_2(t) &= e^{\overline{\lambda} t}\overline{\mathbf{v}} = e^{(\alpha-i\beta)t}(\mathbf{u} - i\mathbf{w}) = e^{\alpha t}[\cos(\beta t) - i\sin(\beta t)](\mathbf{u} - i\mathbf{w}) \\
&= e^{\alpha t}[\cos(\beta t)\mathbf{u} - \sin(\beta t)\mathbf{w}] - i\,e^{\alpha t}[\sin(\beta t)\mathbf{u} + \cos(\beta t)\mathbf{w}].
\end{aligned}$$

Much as in the case of 2$^{\text{nd}}$ order linear homogeneous DEs, we take the following linear combinations of these two solutions as members of our fundamental set instead:

$$\begin{aligned}
\tilde{\mathbf{x}}_1(t) &:= \frac{1}{2}[\mathbf{x}_1(t) + \mathbf{x}_2(t)] = \boxed{e^{\alpha t}[\cos(\beta t)\mathbf{u} - \sin(\beta t)\mathbf{w}]} \\
\tilde{\mathbf{x}}_2(t) &:= \frac{1}{2i}[\mathbf{x}_1(t) - \mathbf{x}_2(t)] = \boxed{e^{\alpha t}[\sin(\beta t)\mathbf{u} + \cos(\beta t)\mathbf{w}]}.
\end{aligned}$$

Example 1: A planar system where the equilibrium at the origin is classified as a **center**

Problem: Consider the 1st order system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ -8 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Write the general solution in such a way that it has no nonreal parts. Then plot the corresponding direction field along with a corresponding phase portrait, and classify the equilibrium at $(0, 0)$.

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Example 2: A planar system where the equilibrium at the origin is classified as a **spiral point**

Problem: Consider the 1st order system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Write the general solution in such a way that it has no nonreal parts. Then plot the corresponding direction field along with a corresponding phase portrait, and classify the equilibrium at $(0, 0)$.

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Example 3:

Problem: Consider the 1st order system

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Write the general solution in such a way that it has no nonreal parts.

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The case when an eigenvalue has algebraic multiplicity > geometric multiplicity

We know how to construct a fundamental set of solutions to the 1^{st} order linear homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with constant coefficients in the case when each eigenvalue of $\mathbf{A}$ has geometric multiplicity equal to its algebraic multiplicity. The problem, when some eigenvalue has geometric multiplicity strictly less than its algebraic multiplicity is that there are not enough *linearly independent* (L.I.) eigenvectors to go with that eigenvalue to fill out its portion of the fundamental set. We investigate this situation next, beginning with a special case. Before doing so, we introduce a couple new matrix-related concepts: the **rank** and **nullity**. For a given matrix $\mathbf{A}$, $\text{rank}(\mathbf{A})$ is the number of linearly independent column vectors it has; $\text{nullity}(\mathbf{A})$ is the difference between the number of columns in $\mathbf{A}$ and its rank. Here are some facts about the rank of a matrix.

Theorem 1: Suppose $\mathbf{A}$ is an m-by-n matrix with complex number entries.

1. $\text{rank}(\mathbf{A}) + \text{nullity}(\mathbf{A}) = n$.
2. Suppose $\mathbf{R}$ is an echelon form of $\mathbf{A}$—i.e., $\mathbf{A}$ can be reduced to $\mathbf{R}$ by means of EROs. Then $\text{rank}(\mathbf{A})$ equals the number of *pivot* columns in $\mathbf{R}$, and $\text{nullity}(\mathbf{A})$ equals the number of *free* columns in $\mathbf{R}$.

3. The value of rank $(\mathbf{A})$ cannot exceed $\min(m, n)$.
4. The number of linearly independent solutions to $\mathbf{A}\mathbf{v} = \mathbf{0}$ equals nullity $(\mathbf{A})$.
5. If $m = n$ (so $\mathbf{A}$ is square), then $\mathbf{A}$ is nonsingular if and only if rank $(\mathbf{A}) = n$ (if and only if nullity $(\mathbf{A}) = 0$).
6. If $m = n$ and λ is an eigenvalue of $\mathbf{A}$, then the geometric multiplicity of λ equals nullity $(\mathbf{A} - \lambda\mathbf{I})$.

An n-by-n matrix for which rank $(\mathbf{A}) < n$ (that is, nullity $(\mathbf{A}) > 0$) is said to be **rank deficient**. Note that the eigenvalues of $\mathbf{A}$ are precisely those complex numbers λ for which $(\mathbf{A} - \lambda\mathbf{I})$ is rank deficient.

Case: geometric multiplicity = 1, algebraic multiplicity = 2

Let us suppose that λ is an eigenvalue of $\mathbf{A}$ whose geometric multiplicity (GM) is 1 while its algebraic multiplicity (AM) is 2. Because GM = 1, we know the collection of eigenvectors corresponding to λ has 1 degree of freedom, so a basis for these eigenvectors consists of just one vector. (Said another way, nullity $(\mathbf{A} - \lambda\mathbf{I}) = 1$.) Let us call this basis eigenvector $\mathbf{v}$. Together, $(\lambda, \mathbf{v})$ give us a solution $\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v}$ to $\mathbf{x}' = \mathbf{A}\mathbf{x}$, and we include it in the construction of a fundamental set of solutions. Given both our experience solving higher-order linear DEs in Chs. 3-4 and the problem from Apr. 6^{th}'s class, we suspect there is another linearly independent solution taking the form $\mathbf{x}(t) = e^{\lambda t}\boldsymbol{\eta} + te^{\lambda t}\mathbf{v}$. We plug this into the 1^{st} order system:

$$\begin{aligned}
\mathbf{x}' = \mathbf{A}\mathbf{x} \quad \text{becomes} \quad & \lambda e^{\lambda t}\boldsymbol{\eta} + e^{\lambda t}\mathbf{v} + \lambda te^{\lambda t}\mathbf{v} = \mathbf{A}(e^{\lambda t}\boldsymbol{\eta} + te^{\lambda t}\mathbf{v}) \\
\Rightarrow \quad & \lambda\boldsymbol{\eta} + \mathbf{v} + \lambda t\mathbf{v} = \mathbf{A}\boldsymbol{\eta} + t\mathbf{A}\mathbf{v} \\
\Rightarrow \quad & \lambda\mathbf{v} = \mathbf{A}\mathbf{v} \quad \text{and} \quad \lambda\boldsymbol{\eta} + \mathbf{v} = \mathbf{A}\boldsymbol{\eta} \\
\Rightarrow \quad & \boxed{(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}} \quad \text{and} \quad (\mathbf{A} - \lambda\mathbf{I})\boldsymbol{\eta} = \mathbf{v}.
\end{aligned}$$

says λ is e-val. and $\vec{v}$ a corresp. e-vector

odd ?

The first of these equations indicates that, if a solution $\mathbf{x}(t)$ of the form we proposed exists, then $\mathbf{v}$ is an eigenvector. It is not obvious that the second equation has a solution but, under the conditions of the scenario we are investigating, it does. (It has infinitely many, in fact, with GM = 1 degree of freedom.) Taking *one* (representative) solution $\boldsymbol{\eta}$, the vector function $\mathbf{x}_2(t) = e^{\lambda t}(\boldsymbol{\eta} + t\mathbf{v})$ solves $\mathbf{x}' = \mathbf{A}\mathbf{x}$ and is linearly independent from others obtained using eigenpairs, making up for the deficiency in our fundamental set construction which occurred because λ had GM = 1 and AM = 2.

Example 4:

Problem: Find the solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$, with $\mathbf{A} = \begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix}$, subject to $\mathbf{x}(0) = (2, -5)$. Draw the phase portrait for this system.

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Case: geometric multiplicity = 1, algebraic multiplicity > 1

Again, we assume λ is an eigenvalue of $\mathbf{A}$ with GM = 1 or, equivalently, that nullity$(\mathbf{A}) = 1$. Let $\mathbf{v}$ be a corresponding eigenvector. As we handled the case where AM = 2 above, we assume here that AM = $k > 2$ so that, along with $e^{\lambda t}\mathbf{v}$, we must find $k-1$ additional solutions associated somehow with λ to be included in our construction of a fundamental set of solutions to $\mathbf{x}' = \mathbf{A}\mathbf{x}$. As before, we look for a solution of the form $\mathbf{x}(t) = e^{\lambda t}(\boldsymbol{\eta} + t\mathbf{v})$, which requires that we solve $(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{\eta} = \mathbf{v}$. Since nullity$(\mathbf{A} - \lambda\mathbf{I}) = 1$, there is just one degree of freedom in the collection of vectors $\boldsymbol{\eta}$ that solve this problem, which means this process can give us just one additional entry for our fundamental set. The key is that we will need to take this process up k levels. At level 1, we find a representative eigenvector $\mathbf{v}$. At level 2, we solve for a vector $\boldsymbol{\eta}^{(1)}$ in $\mathbb{R}^n$ that satisfies $(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{\eta} = \mathbf{v}$. At level 3, with $\boldsymbol{\eta}^{(1)}$ already fixed, we solve for $\boldsymbol{\eta}^{(2)}$, and so on. This is summarized in the table below.

Level	Matrix Problem to Be Solved	Resulting Solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$
1	$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$	$e^{\lambda t}\mathbf{v}$
2	$(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{\eta}^{(1)} = \mathbf{v}$	$e^{\lambda t}(\boldsymbol{\eta}^{(1)} + t\mathbf{v})$
3	$(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{\eta}^{(2)} = \boldsymbol{\eta}^{(1)}$	$e^{\lambda t}\left(\boldsymbol{\eta}^{(2)} + t\boldsymbol{\eta}^{(1)} + \frac{t^2}{2!}\mathbf{v}\right)$
4	$(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{\eta}^{(3)} = \boldsymbol{\eta}^{(2)}$	$e^{\lambda t}\left(\boldsymbol{\eta}^{(3)} + t\boldsymbol{\eta}^{(2)} + \frac{t^2}{2!}\boldsymbol{\eta}^{(1)} + \frac{t^3}{3!}\mathbf{v}\right)$
$\vdots$	$\vdots$	$\vdots$
k	$(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{\eta}^{(k-1)} = \boldsymbol{\eta}^{(k-2)}$	$e^{\lambda t}\left(\boldsymbol{\eta}^{(k-1)} + t\boldsymbol{\eta}^{(k-2)} + \cdots + \frac{t^{k-2}}{(k-2)!}\boldsymbol{\eta}^{(1)} + \frac{t^{k-1}}{(k-1)!}\mathbf{v}\right)$

Example 5: After Exercise 17, Section 7.8

Problem: Find the general solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ when $\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix}$.

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A word about the other cases

We have been discussing cases in which **A** has an eigenvalue whose GM $<$ AM. We have stuck to instances in which GM = 1. There are numerous ways in which one might encounter 1 $<$ GM $<$ AM, and these are more complicated. We will illustrate the new wrinkles that appear in such cases with an example, and leave the rest as a topic of exploration in an *advanced* course in ODEs.

Example 6: After Exercise 18, Section 7.8

Problem: Find the general solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ when $\mathbf{A} = \begin{pmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{pmatrix}$.

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Ex.

$$\vec{x}' = \begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix} \vec{x}$$

Facts: $\lambda = 5$ is only e-value (AM = 2)

But RREF$(A - 5I)$ has just one free col.

$\implies$ GM = 1

basis e-vector for eigenspace $E_5 = \text{null}(A - 5I)$: $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

So $e^{5t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is a solution of system

But, since $\lambda = 5$ had AM = 2, we hoped it would contribute a 2nd solution, L.I. from this one.

There is a 2^{nd} solution. It has form

$$\underbrace{e^{5t}\begin{bmatrix} \\ \\ \end{bmatrix}}_{\text{some vector } \vec{\eta}} + \underbrace{te^{5t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}}_{\text{already found this e-vector}}$$

Note: Found $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ by solving

$$(A - 5I)\vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ — i.e., augmenting w/ $\vec{0}$ vector

$$\left[\begin{array}{c|c} A - 5I & \begin{matrix} 0 \\ 0 \end{matrix} \end{array}\right]$$

We find $\vec{\eta}$ by taking any solution of $(A - 5I)\vec{\eta} = \vec{v}$

that is, we augment

now augment with $\vec{v}$

$$\left[\begin{array}{c|c} A - 5I & \begin{matrix} 1 \\ -2 \end{matrix} \end{array}\right]$$

$$A = \begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix} \quad \text{so} \quad A - 5I = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$$

Don't need to augment

$$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ -4 & -2 & 0 \end{array}\right]$$ — we know already there will be one free column, get a basis vector of $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Do need to augment

$$\left[\begin{array}{cc|c} 2 & 1 & 1 \\ -4 & -2 & -2 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 1/2 & 1/2 \\ \underbrace{0 \quad\quad 0}_{} & & \!\!\!\!\!\!\underbrace{0}_{\text{must happen}} \end{array}\right]$$

Components

$\vec{\eta} = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$ must satisfy $\eta_1 + \frac{1}{2}\eta_2 = \frac{1}{2}$

Goal here:

Not to describe all possible η

Rather to find one solution $\vec{\eta}$

So I can choose any $\vec{\eta}$ so its components satisfy the requirement.

$$\eta = \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix},}_{\text{choose one}}$$

2^{nd} solution:

$$e^{5t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + te^{5t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

General soln (reflecting 2 degrees of freedom - needed so ICs can be met) is

$$\vec{x}(t) = c_1 e^{5t}\begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{5t}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} + t\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right)$$

$$= \Phi(t) \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

with

$$\Phi(t) = \begin{bmatrix} e^{5t} & e^{5t} + te^{5t} \\ -2e^{5t} & -e^{5t} - 2te^{5t} \end{bmatrix}$$