The case when an eigenvalue has algebraic multiplicity > geometric multiplicity

We know how to construct a fundamental set of solutions to the 1^{st} order linear homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with constant coefficients in the case when each eigenvalue of $\mathbf{A}$ has geometric multiplicity equal to its algebraic multiplicity. The problem, when some eigenvalue has geometric multiplicity strictly less than its algebraic multiplicity is that there are not enough *linearly independent* (L.I.) eigenvectors to go with that eigenvalue to fill out its portion of the fundamental set. We investigate this situation next, beginning with a special case. Before doing so, we introduce a couple new matrix-related concepts: the **rank** and **nullity**. For a given matrix $\mathbf{A}$, $\operatorname{rank}(\mathbf{A})$ is the number of linearly independent column vectors it has; $\operatorname{nullity}(\mathbf{A})$ is the difference between the number of columns in $\mathbf{A}$ and its rank. Here are some facts about the rank of a matrix.

Theorem 1: Suppose $\mathbf{A}$ is an m-by-n matrix with complex number entries.

1. $\operatorname{rank}(\mathbf{A}) + \operatorname{nullity}(\mathbf{A}) = n$.
2. Suppose $\mathbf{R}$ is an echelon form of $\mathbf{A}$—i.e., $\mathbf{A}$ can be reduced to $\mathbf{R}$ by means of EROs. Then $\operatorname{rank}(\mathbf{A})$ equals the number of *pivot* columns in $\mathbf{R}$, and $\operatorname{nullity}(\mathbf{A})$ equals the number of *free* columns in $\mathbf{R}$.
3. The value of $\operatorname{rank}(\mathbf{A})$ cannot exceed $\min(m, n)$.
4. The number of linearly independent solutions to $\mathbf{A}\mathbf{v} = \mathbf{0}$ equals $\operatorname{nullity}(\mathbf{A})$.
5. If $m = n$ (so $\mathbf{A}$ is square), then $\mathbf{A}$ is nonsingular if and only if $\operatorname{rank}(\mathbf{A}) = n$ (if and only if $\operatorname{nullity}(\mathbf{A}) = 0$).
6. If $m = n$ and λ is an eigenvalue of $\mathbf{A}$, then the geometric multiplicity of λ equals $\operatorname{nullity}(\mathbf{A} - \lambda\mathbf{I})$.

An n-by-n matrix for which $\operatorname{rank}(\mathbf{A}) < n$ (that is, $\operatorname{nullity}(\mathbf{A}) > 0$) is said to be **rank deficient**. Note that the eigenvalues of $\mathbf{A}$ are precisely those complex numbers λ for which $(\mathbf{A} - \lambda\mathbf{I})$ is rank deficient.

Case: geometric multiplicity = 1, algebraic multiplicity = 2

Let us suppose that λ is an eigenvalue of $\mathbf{A}$ whose geometric multiplicity (GM) is 1 while its algebraic multiplicity (AM) is 2. Because GM = 1, we know the collection of eigenvectors corresponding to λ has 1 degree of freedom, so a basis for these eigenvectors consists of just one vector.

(Said another way, nullity $(\mathbf{A} - \lambda\mathbf{I}) = 1$.) Let us call this basis eigenvector $\mathbf{v}$. Together, $(\lambda, \mathbf{v})$ give us a solution $\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v}$ to $\mathbf{x}' = \mathbf{A}\mathbf{x}$, and we include it in the construction of a fundamental set of solutions. Given both our experience solving higher-order linear DEs in Chs. 3-4 and the problem from Apr. 6th's class, we suspect there is another linearly independent solution taking the form $\mathbf{x}(t) = e^{\lambda t}\boldsymbol{\eta} + te^{\lambda t}\mathbf{v}$. We plug this into the 1st order system:

$$\begin{array}{rcrcl} \mathbf{x}' = \mathbf{A}\mathbf{x} & \text{becomes} & \lambda e^{\lambda t}\boldsymbol{\eta} + e^{\lambda t}\mathbf{v} + \lambda t e^{\lambda t}\mathbf{v} & = & \mathbf{A}(e^{\lambda t}\boldsymbol{\eta} + te^{\lambda t}\mathbf{v}) \\ & \Rightarrow & \lambda\boldsymbol{\eta} + \mathbf{v} + \lambda t\mathbf{v} & = & \mathbf{A}\boldsymbol{\eta} + t\mathbf{A}\mathbf{v} \\ & \Rightarrow & \lambda\mathbf{v} = \mathbf{A}\mathbf{v} \quad \text{and} \quad \lambda\boldsymbol{\eta} + \mathbf{v} & = & \mathbf{A}\boldsymbol{\eta} \\ & \Rightarrow & (\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0} \quad \text{and} \quad (\mathbf{A} - \lambda\mathbf{I})\boldsymbol{\eta} & = & \mathbf{v}. \end{array}$$

The first of these equations indicates that, if a solution $\mathbf{x}(t)$ of the form we proposed exists, then $\mathbf{v}$ is an eigenvector. It is not obvious that the second equation has a solution but, under the conditions of the scenario we are investigating, it does. (It has infinitely many, in fact, with GM = 1 degree of freedom.) Taking *one* (representative) solution $\boldsymbol{\eta}$, the vector function $\mathbf{x}_2(t) = e^{\lambda t}(\boldsymbol{\eta} + t\mathbf{v})$ solves $\mathbf{x}' = \mathbf{A}\mathbf{x}$ and is linearly independent from others obtained using eigenpairs, making up for the deficiency in our fundamental set construction which occurred because λ had GM = 1 and AM = 2.

Example 4:

Problem: Find the solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$, with $\mathbf{A} = \begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix}$, subject to $\mathbf{x}(0) = (2, -5)$. Draw the phase portrait for this system.

■

Case: $\underbrace{\textbf{geometric multiplicity = 1}}$, algebraic multiplicity > 1

Again, we assume λ is an eigenvalue of $\mathbf{A}$ with GM = 1 or, equivalently, that nullity $(\mathbf{A}) = 1$. Let $\mathbf{v}$ be a corresponding eigenvector. As we handled the case where AM = 2 above, we assume here that AM = $k > 2$ so that, along with $e^{\lambda t}\mathbf{v}$, we must find $k - 1$ additional solutions associated somehow with λ to be included in our construction of a fundamental set of solutions to $\mathbf{x}' = \mathbf{A}\mathbf{x}$. As before, we look for a solution of the form $\mathbf{x}(t) = e^{\lambda t}(\boldsymbol{\eta} + t\mathbf{v})$, which requires that we solve $(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{\eta} = \mathbf{v}$. Since nullity $(\mathbf{A} - \lambda\mathbf{I}) = 1$, there is just one degree of freedom in the collection of vectors $\boldsymbol{\eta}$ that solve this problem, which means this process can give us just one additional entry for our fundamental set. The key is that we will need to take this process up k levels. At level 1, we find a representative eigenvector $\mathbf{v}$. At level 2, we solve for a vector $\boldsymbol{\eta}^{(1)}$ in $\mathbb{R}^n$ that satisfies $(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{\eta} = \mathbf{v}$. At level 3, with $\boldsymbol{\eta}^{(1)}$ already fixed, we solve for $\boldsymbol{\eta}^{(2)}$, and so on. This is summarized in the table below.

Level	Matrix Problem to Be Solved	Resulting Solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$
1	$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$	$e^{\lambda t}\mathbf{v}$
2	$(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{\eta}^{(1)} = \mathbf{v}$	$e^{\lambda t}(\boldsymbol{\eta}^{(1)} + t\mathbf{v})$
3	$(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{\eta}^{(2)} = \boldsymbol{\eta}^{(1)}$	$e^{\lambda t}\left(\boldsymbol{\eta}^{(2)} + t\boldsymbol{\eta}^{(1)} + \frac{t^2}{\boxed{2!}}\mathbf{v}\right)$
$\underline{4}$	$(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{\eta}^{(3)} = \boldsymbol{\eta}^{(2)}$	$e^{\lambda t}\left(\boldsymbol{\eta}^{(3)} + t\boldsymbol{\eta}^{(2)} + \frac{t^2}{2!}\boldsymbol{\eta}^{(1)} + \frac{t^3}{\underline{3!}}\mathbf{v}\right)$
⋮	⋮	⋮
k	$(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{\eta}^{(k-1)} = \boldsymbol{\eta}^{(k-2)}$	$e^{\lambda t}\left(\boldsymbol{\eta}^{(k-1)} + t\boldsymbol{\eta}^{(k-2)} + \cdots + \frac{t^{k-2}}{(k-2)!}\boldsymbol{\eta}^{(1)} + \frac{t^{k-1}}{(k-1)!}\mathbf{v}\right)$

Example 5: After Exercise 17, Section 7.8

Problem: Find the general solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ when $\mathbf{A} = \boxed{\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix}}$.

$$\det(A - \lambda I) = -(\lambda - 2)^3$$

GM = 1, basis e-vector: $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \vec{v}$ — solves $(A - 2I)\vec{v} = \vec{0}$

■

A word about the other cases

We have been discussing cases in which **A** has an eigenvalue whose GM < AM. We have stuck to instances in which GM = 1. There are numerous ways in which one might encounter 1 < GM < AM, and these are more complicated. We will illustrate the new wrinkles that appear in such cases with an example, and leave the rest as a topic of exploration in an *advanced* course in ODEs.

Example 6: After Exercise 18, Section 7.8

$$AM > GM > 1$$

Problem: Find the general solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ when $\mathbf{A} = \underbrace{\begin{pmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{pmatrix}}$.

■

Won't treat this sort of problem this semester

1st Order Linear Systems: Loose Ends

For A, have one soln. to $\vec{x}' = Ax$ that comes from eigenpairs:

$$e^{2t}\begin{bmatrix}0\\1\\-1\end{bmatrix}$$

(Serve as one column in fund'l matrix $\Phi(t)$, but need 3)

From Monday, learned a 2nd soln. exists of form

$$e^{2t}\vec{\eta} + te^{2t}\begin{bmatrix}0\\1\\-1\end{bmatrix}$$

and $\vec{\eta}$ must satisfy

$$(A-2I)\underset{\substack{\uparrow\\ \text{solve for}\\ \vec{\eta}}}{\vec{\eta}} = \underset{\vec{v}\ \text{(eigenvector)}}{\begin{bmatrix}0\\1\\-1\end{bmatrix}}$$

$$\underbrace{\begin{bmatrix}-1 & 1 & 1\\ 2 & -1 & -1\\ -3 & 2 & 2\end{bmatrix}}_{A-2I}\left|\ \overset{\substack{\text{augment}\\ \text{with } \vec{v}\\ \downarrow}}{\begin{matrix}0\\1\\-1\end{matrix}}\right] \rightsquigarrow \left[\begin{array}{ccc|c}1 & 0 & 0 & 1\\ 0 & 1 & 1 & 1\\ 0 & 0 & 0 & 0\end{array}\right]\begin{matrix}\to \eta_1 = 1\\ \to \boxed{\eta_2+\eta_3=1}\\ \end{matrix}$$

$$\vec{\eta} = \begin{bmatrix}1\\0\\1\end{bmatrix} \longrightarrow \text{2nd soln. } e^{2t}\left(\begin{bmatrix}1\\0\\1\end{bmatrix} + t\begin{bmatrix}0\\1\\-1\end{bmatrix}\right)$$

can serve as a 2nd col. in $\Phi(t)$.

Still need a 3rd col.

Since $\lambda = 2$ is still contributing fewer cols. to $\Phi(t)$ than its AM says it should, we may find a 3rd now of the form

$$e^{2t}\left(\vec{w} + \overset{\vec{\eta}\, t}{t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}} + \overset{t^2 \vec{v}}{\frac{1}{2!} t^2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}} \right)$$

and require $\vec{w}$ satisfies $$\left(A - 2I\right)\vec{w} = \vec{\eta}$$

Use GE:

$$\left[\begin{array}{ccc|c} & & & \text{augment with } \vec{\eta} \\ -1 & 1 & 1 & 1 \\ 2 & -1 & -1 & 0 \\ -3 & 2 & 2 & 1 \end{array}\right] \hookrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\underbrace{}_{A-2I}$$

says $w_1 = 1$

$w_2 + w_3 = 2$

Can take

$$\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Now have 3rd soln./col. for Φ: $$e^{2t}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{2} t^2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right)$$

General linear (possibly nonhomogeneous) problems

The most general linear 1$^{\text{st}}$ order system of DEs takes the form

$$\boxed{\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t).} \tag{1}$$

– still will stick to case $A(t) = A$

That is, the coefficients of terms involving the components $x_1, \ldots, x_n$ of $\mathbf{x}$ may be time-dependent (resulting in a matrix $\mathbf{A} = \mathbf{A}(t)$ which is time-dependent), and the problem under consideration may not be homogeneous (resulting in a nonzero $\mathbf{f}(t)$ term). While it is not the only possibly route to a solution, it is possible to attack the system (1) using the same scheme as in Chs. 3–4:

- First solve the complementary homogeneous problem: $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$. Of course, this is not necessarily an easy problem. Assuming $\mathbf{A}$ is n-by-n, it involves building a fundamental set of solutions $\{\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)\}$. We have studied how to do this in the case that

  (i) $\mathbf{A}(t) = \mathbf{A}$ is a constant matrix, and

  (ii) the geometric multiplicity of any eigenvalue λ of $\mathbf{A}$ is either equal to its algebraic multiplicity, or is 1.

  However one obtains this fundamental set of solutions, we write the homogeneous solution as $\mathbf{x}_h(t) \;=\; c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t)$.

- Find a particular solution $\mathbf{x}_p(t)$ of (1) using one of the methods we studied in Chs. 3–4: the method of undetermined coefficients, or variation of parameters, adapted to the vector setting.

- Obtain the general solution of (1) by putting these two together: $\mathbf{x}(t) = \mathbf{x}_h(t) + \mathbf{x}_p(t)$.

As there is nothing very different here from Ch. 3, we will not probe these issues further. The interested reader may consult Sections 7.7 and 7.9.

Variation of Parameters for 1$^{\text{st}}$ order linear systems

In solving the problem (1), the first step is to find a fundamental set of solutions to the related homogeneous problem. We assume such a set has been found, and have been assembled into a fundamental matrix $\mathbf{\Phi}(t)$, so that the general solution of the corresponding homogeneous problem is $\mathbf{x}_h(t) = \mathbf{\Phi}(t)\mathbf{c}$, where $\mathbf{c}$ is a vector of arbitrary constants (i.e., a constant vector). As before, the underlying idea of **variation of parameters** is to search for a particular solution with a vector function $\mathbf{v}(t)$ in place of the constant vector $\mathbf{c}$:

$$\mathbf{x}_p(t) \;=\; \mathbf{\Phi}(t)\mathbf{v}(t).$$

Recall from Chapter 2: Scalar problems

$$y' = a(t)\, y + f(t)$$

had solns

$$y(t) = y_h(t) + y_p(t)$$

where $y_h(t) = c\,\varphi(t)$

and $$\boxed{y_p(t) = \varphi(t) \cdot \int \frac{f(t)}{\varphi(t)}\, dt}$$ Variation of Parameters formula

In Ch. 3, dealing w/ a system (still 1st-order, linear)

$$\frac{d\vec{x}}{dt} = A\vec{x} + \vec{f}(t)$$

Still true: Soln (general soln.) is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

where

$$\vec{x}_h(t) = \Phi(t)\, \vec{c}$$ (what we've been finding for last wk in homog. probs.)

and $$\boxed{\vec{x}_p(t) = \Phi(t) \int \Phi^{-1}(t)\, \vec{f}(t)\, dt}$$ Variation of params formula

The derivative of this proposed $\mathbf{x}_p(t)$

$$\frac{d}{dt}\Big[\mathbf{\Phi}(t)\mathbf{v}(t)\Big] \;=\; \mathbf{\Phi}'(t)\mathbf{v}(t) + \mathbf{\Phi}(t)\mathbf{v}'(t).$$

Inserting this derivative along with the proposed $\mathbf{x}_p(t)$ into (1), we have

$$\mathbf{\Phi}'(t)\mathbf{v}(t) + \mathbf{\Phi}(t)\mathbf{v}'(t) \;=\; \mathbf{A}(t)\mathbf{\Phi}(t)\mathbf{v}(t) + \mathbf{f}(t),$$

which can be rearranged to say

$$\mathbf{\Phi}(t)\mathbf{v}'(t) \;=\; \mathbf{A}(t)\mathbf{\Phi}(t)\mathbf{v}(t) - \mathbf{\Phi}'(t)\mathbf{v}(t) + \mathbf{f}(t) \;=\; \left[\mathbf{A}(t)\mathbf{\Phi}(t) - \mathbf{\Phi}'(t)\right]\mathbf{v}(t) + \mathbf{f}(t) \;=\; \mathbf{f}(t).$$

The crucial step uses the fact that $\mathbf{A}(t)\mathbf{\Phi}(t) - \mathbf{\Phi}'(t) = \mathbf{0}$, which holds because all the columns of $\mathbf{\Phi}(t)$ satisfy $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$. Working on the end result $\mathbf{\Phi}(t)\mathbf{v}'(t) = \mathbf{f}(t)$, we get

$$\mathbf{v}'(t) \;=\; \mathbf{\Phi}^{-1}(t)\mathbf{f}(t), \qquad \text{or} \qquad \mathbf{v}(t) \;=\; \int \mathbf{\Phi}^{-1}(t)\mathbf{f}(t)\,dt.$$

There is no doubt the fundamental matrix $\mathbf{\Phi}(t)$ is nonsingular, and we can complete the **Variation of Parameters Formula** for a particular solution:

$$\boxed{\mathbf{x}_p(t) \;=\; \mathbf{\Phi}(t)\int \mathbf{\Phi}^{-1}(t)\mathbf{f}(t)\,dt. \qquad (2)}$$

This formula for $\mathbf{x}_p$ involving vector functions completely mirrors the scalar Variation of Parmaters formula from Chapter 2. However, it might be advisable to find $\mathbf{x}_p$ following these steps:

1. First solve $\mathbf{\Phi}(t)\mathbf{v}' = \mathbf{f}(t)$, most likely using Cramer's Rule, for $\mathbf{v}'(t)$. (In class, I called this $\mathbf{v}'$ by the name $\mathbf{u}$, instead.)
2. Integrate $\mathbf{v}'(t)$ to get $\mathbf{v}(t)$, taking the arbitrary constant of integration to be zero. Compute the product $\mathbf{x}_p(t) = \mathbf{\Phi}(t)\mathbf{v}(t)$.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad \vec{f}(t) = \begin{bmatrix} 2e^{4t} \\ e^{4t} \end{bmatrix}$$

Example 7:

Find the general solution of

$$\left.\begin{aligned} x' &= x + 2y + 2e^{4t} \\ y' &= 2x + y + e^{4t} \end{aligned}\right\}$$

Define $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$

In matrix form $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}$, the system is

$$\frac{d}{dt}\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 2e^{4t} \\ e^{4t} \end{bmatrix}.$$

So $\frac{d}{dt}\vec{x} = \begin{bmatrix} x' \\ y' \end{bmatrix}$

The eigenpairs (eigenvalue with corresponding basis eigenvector) of the matrix are

$$-1, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \text{and} \qquad 3, \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

Homogeneous version of this problem: $\frac{d}{dt}\vec{x} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\vec{x}$

has solns. $e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

So

$$\vec{x}_h(t) = \underbrace{\begin{bmatrix} e^{-t} & e^{3t} \\ -e^{-t} & e^{3t} \end{bmatrix}}_{\Phi(t)} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Recall Inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Wronskian $= \det(\Phi(t)) = e^{2t} - (-e^{2t}) = 2e^{2t}$

$\Rightarrow$ Inverse of $\Phi(t)$ (since 2 ÷ by −2) is

$$\Phi^{-1}(t) = \frac{1}{2e^{2t}} \cdot \begin{bmatrix} e^{3t} & -e^{3t} \\ e^{-t} & e^{-t} \end{bmatrix}$$

In Var. of params. formula:

$$\vec{v}\,'(t) = \Phi^{-1}(t)\,\vec{f}(t) = \frac{1}{2e^{2t}}\begin{bmatrix} e^{3t} & -e^{3t} \\ e^{-t} & e^{-t} \end{bmatrix}\begin{bmatrix} 2e^{4t} \\ e^{4t} \end{bmatrix}$$

$$= \frac{1}{2e^{2t}}\left(2e^{4t}\begin{bmatrix} e^{3t} \\ e^{-t} \end{bmatrix} + e^{4t}\begin{bmatrix} -e^{3t} \\ e^{-t} \end{bmatrix}\right)$$

corrected exponents

$$= \frac{1}{2e^{2t}} \left(\begin{bmatrix} 2e^{7t} \\ 2e^{3t} \end{bmatrix} + \begin{bmatrix} -e^{7t} \\ e^{3t} \end{bmatrix} \right) = \begin{bmatrix} \frac{1}{2} e^{5t} \\ \frac{3}{2} e^{t} \end{bmatrix} = \begin{bmatrix} v_1'(t) \\ v_2'(t) \end{bmatrix}$$

We integrate this:

$$\vec{v}(t) = \int \Phi^{-1}(t) \vec{f}(t) \, dt = \begin{bmatrix} \int 1/2 \, e^{5t} \, dt \\ \int 3/2 \, e^{t} \, dt \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{10} e^{5t} \\ \frac{3}{2} e^{t} \end{bmatrix} = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}$$

Finally

$$\vec{x}_p(t) = \Phi(t) \cdot \int \Phi^{-1}(t) \vec{f}(t) \, dt = \Phi(t) \vec{v}(t)$$

$$= \underbrace{\begin{bmatrix} e^{-t} & e^{3t} \\ -e^{-t} & e^{3t} \end{bmatrix}} \begin{bmatrix} \frac{1}{10} e^{5t} \\ \frac{3}{2} e^{t} \end{bmatrix},$$

a 2×1 vector (time ran out, but see "prepared" answer on the next page)

So gen'l soln.

$$\vec{x}(t) = \Phi(t) \vec{c} + \vec{x}_p(t)$$

Example 7 continued

The ~~yielding a~~ fundamental matrix is

$$\mathbf{\Phi}(t) = \begin{bmatrix} e^{-t} & e^{3t} \\ -e^{-t} & e^{3t} \end{bmatrix} \qquad \text{with Wronskian} \qquad \begin{vmatrix} e^{-t} & e^{3t} \\ -e^{-t} & e^{3t} \end{vmatrix} = 2e^{2t}.$$

The components v_1', v_2' of $\mathbf{v}' = d\mathbf{v}/dt$, obtained using Cramer's Rule, are

$$v_1' = \frac{1}{2e^{2t}} \begin{vmatrix} 2e^{4t} & e^{3t} \\ e^{4t} & e^{3t} \end{vmatrix} = \frac{1}{2} e^{5t}, \qquad \text{and} \qquad v_2' = \frac{1}{2e^{2t}} \begin{vmatrix} e^{-t} & 2e^{4t} \\ -e^{-t} & e^{4t} \end{vmatrix} = \frac{3}{2} e^{t},$$

so

$$v_1(t) = \int \frac{1}{2} e^{5t}\, dt = \frac{1}{10} e^{5t}, \qquad v_2(t) = \int \frac{3}{2} e^{t}\, dt = \frac{3}{2} e^{t},$$

and

$$\mathbf{x}_p(t) = \mathbf{\Phi}(t)\mathbf{v}(t) = \begin{bmatrix} e^{-t} & e^{3t} \\ -e^{-t} & e^{3t} \end{bmatrix} \begin{bmatrix} (1/10)e^{5t} \\ (3/2)e^{t} \end{bmatrix} = \frac{1}{10} e^{5t} \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix} + \frac{3}{2} e^{t} \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix}$$

$$= \begin{bmatrix} (8/5)e^{4t} \\ (7/5)e^{4t} \end{bmatrix}.$$

■

Phase portraits for 2-by-2 systems

We consider here the linear homogeneous problem $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = (a_{ij})$ is a (constant) 2-by-2 matrix. We will write the components of $\mathbf{x}$ as x and y.

In two dimensions, the phase portrait of a 1st order linear homogeneous system of DEs usually goes hand in hand with the classification of equilibria, so the discussions are linked. An **equilibrium point**, you will recall, is any **state** (or tuple of dependent variables) $\mathbf{x} = (x, y)$ at which the dependent variables simultaneously have zero derivatives—that is, any point (x, y) which satisfies

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix}.$$

So, when $\det(\mathbf{A}) \neq 0$, the only solution to this equation (and hence the only equilibrium point) is $\mathbf{x} = (0, 0)$. As it is usually the case for most problems, we will assume, for now, that $\det(\mathbf{A}) \neq 0$. Of course, this means the eigenvalues of $\mathbf{A}$ are nonzero.

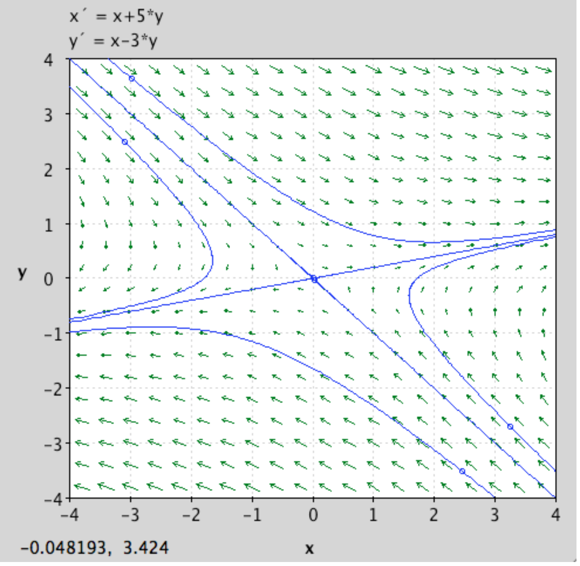
We have discussed some of the most important classifications of the equilibrium at the origin, along with their accompanying phase portraits. We catalog these here, along with mentioning some new ones. In each case, I have used a phase portrait app for autonomous 1st order systems like the one used in class to obtain a direction field and phase portrait, and we identify the features noted in the eigenpairs of the matrix which dictate the behavior we see in the phase portrait.

1. Origin is a **saddle point**

   Consider the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where

   $$\mathbf{A} = \begin{pmatrix} 1 & 5 \\ 1 & -3 \end{pmatrix}.$$

   The direction field and phase portrait is pictured at right.
   Analyzing this matrix, we find it has eigenpairs



eigenvalue	basis eigenvector(s)
2	(5, 1)
-4	(-1, 1)

   yielding general solution

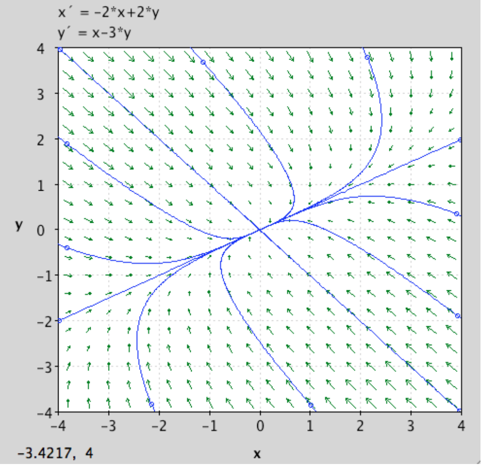
   $$\mathbf{x}(t) \;=\; c_1 e^{2t}\begin{pmatrix} 5 \\ 1 \end{pmatrix} + c_2 e^{-4t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

   The eigenvectors explain the straight lines through the origin. These lines are, in fact, four separate trajectories: one arising when $c_1 = 0, c_2 > 0$; a second when $c_1 = 0, c_2 < 0$ (these two tend toward the origin as $t \to \infty$ because of the sign of the eigenvalue (-4)); a third when $c_1 > 0, c_2 = 0$; a fourth when $c_1 < 0, c_2 = 0$ (these two tend toward the origin as $t \to -\infty$).

   A saddle point occurs whenever the eigenvalues of the 2-by-2 matrix **A** are real and of opposite sign. When you sketch a phase portrait, your drawing should include arrows on trajectories indicating direction of flow for increasing time. Make sure you are able to identify *eight* trajectories on the picture here, and know the appropriate orientation (arrow directions) on all eight.

2. Origin is a **node**

   The term **node** is applied to all situations in which both eigenvalues are real and of the same sign. But there are several kinds of nodes.



Node: Consider the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where

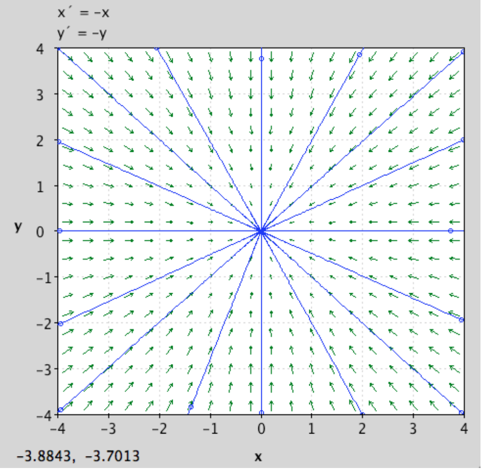
$$\mathbf{A} = \begin{pmatrix} -2 & 2 \\ 1 & -3 \end{pmatrix}.$$

This matrix has eigenpairs

eigenvalue	basis eigenvector(s)
-1	(2, 1)
-4	(-1, 1)

yielding general solution

$$\mathbf{x}(t) \;=\; c_1 e^{-t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{-4t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

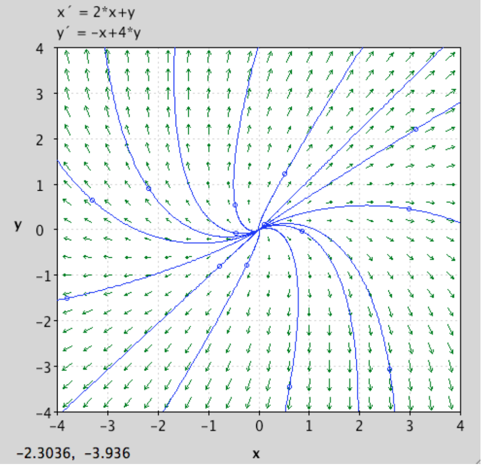


Proper Node: Consider the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where

$$\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Obviously the eigenvalue (-1) has AM = 2. It is easily shown that GM = 2, and a basis of eigenvectors is $\{(1,0),(0,1)\}$, yielding general solution

$$\mathbf{x}(t) = c_1 e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$



Improper/Degenerate Node: Consider the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}.$$

The feature of $\mathbf{A}$ which may be used to identify a **degenerate node** is that it has a real eigenvalue (here, it is 3) with AM = 2, but GM = 1. In this instance a basis eigenvector is $\mathbf{v} = (1,1)$, and one solution of $(\mathbf{A} - 3\mathbf{I})\boldsymbol{\eta} = \mathbf{v}$ is $\boldsymbol{\eta} = (-1,0)$, yielding general solution

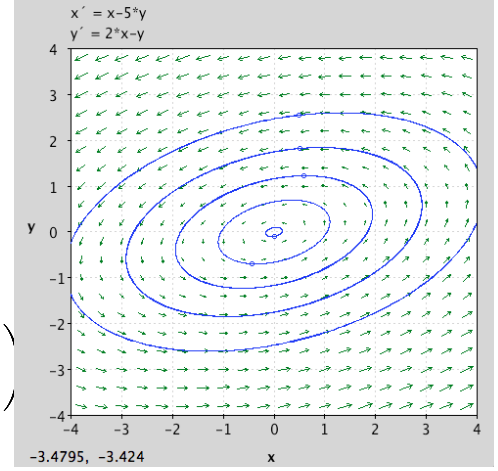
$$\mathbf{x}(t) = c_1 e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{3t}\left[\begin{pmatrix} -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right].$$

For solution trajectories with $c_2 = 0$, we get two rays emanating from the origin along the direction of the line parallel to the vector $(1,1)$. For those with $c_2 \neq 0$, notice that, as $t \to \pm\infty$, the relative influence of the two vectors in the sum

$$(-1,0)+t(1,1)$$

will be heavily tilted toward the eigenvector $(1,1)$. This means that, for $|t|$ large, trajectories should be more and more parallel to the vector $(1,1)$ as $t \to \pm\infty$, but during some intermediate range of t-values, it has to turn 180°.

3. Origin is a **center**



Consider the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where

$$\mathbf{A} = \begin{pmatrix} 1 & -5 \\ 2 & -1 \end{pmatrix}.$$

Here, the eigenvalues are $\pm 3i$, and to the eigenvalue $3i$ there is a corresponding eigenvector $(-5,-1+3i)$. This yields general soln

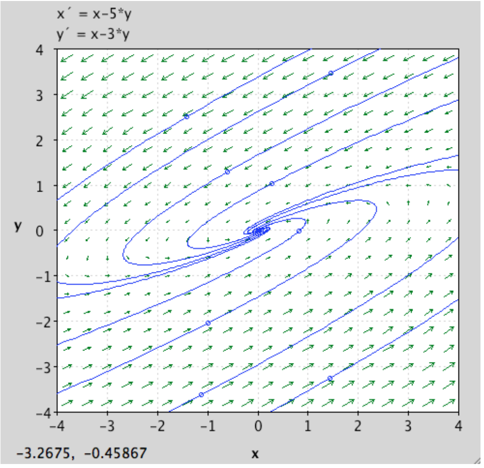
$$\mathbf{x}(t) \;=\; c_1 \begin{pmatrix} -5\cos(3t) \\ -\cos(3t) - 3\sin(3t) \end{pmatrix} + c_2 \begin{pmatrix} -5\sin(3t) \\ -\sin(3t) + 3\cos(3t) \end{pmatrix}$$

Clearly there is a periodic nature to these solutions, explaining the closed loop trajectories. To determine orientation (direction of "flow" as t increases), take a test point, say, $(1,0)$. At this point, we have rate of change

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & -5 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

showing that, when we are at the point $(1,0)$, flow is upward to the right. Once you draw an arrow to this effect, orientation along any trajectory is the same.

4. Origin is a **spiral point**



Consider the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where

$$\mathbf{A} = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix}.$$

This matrix has eigenvalue $\lambda = -1 + i$ with corresponding eigenvector $(5, 2-i)$, yielding general soln

$$\mathbf{x}(t) \;=\; c_1 e^{-t} \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 5\sin t \\ 2\sin t - \cos t \end{pmatrix}.$$

If it weren't for the presence of the factor e^{-t}, one would expect another *center*—trajectories forming closed loops. But, because of the exponential decay scaling factor, we have trajectories that spiral inward (note how you would add arrows to indicate orientation) toward the origin.

Stability diagram

For 2-by-2 matrix $\mathbf{A} = (a_{ij})$, let us define

$$\begin{aligned} \tau &= \operatorname{trace}(\mathbf{A}) &= a_{11} + a_{12}, \\ \Delta &= \det(\mathbf{A}) &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}. \end{aligned}$$

Notice that the characteristic polynomial of $\mathbf{A}$, in this 2-by-2 case, is

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = \lambda^2 - (a_{11} + a_{12})\lambda + a_{11}a_{22} - a_{21}a_{12} = \lambda^2 - \tau\lambda + \Delta,$$

which has roots (the eigenvalues of $\mathbf{A}$)

$$\lambda_{1,2} = \frac{\tau}{2} \pm \frac{1}{2}\sqrt{\tau^2 - 4\Delta}.$$

Saddle points arise when are the two eigenvalues are nonzero real numbers of opposite sign, and this occurs precisely when

$$\sqrt{\tau^2 - 4\Delta} > |\tau| \qquad \Leftrightarrow \qquad \Delta < 0.$$

Nodes arise when these eigenvalues are distinct, but of the same sign, and this occurs precisely when the expression under the radical

$$0 < \tau^2 - 4\Delta < \tau^2 \qquad \Leftrightarrow \qquad 0 < \Delta < \tau^2/4.$$

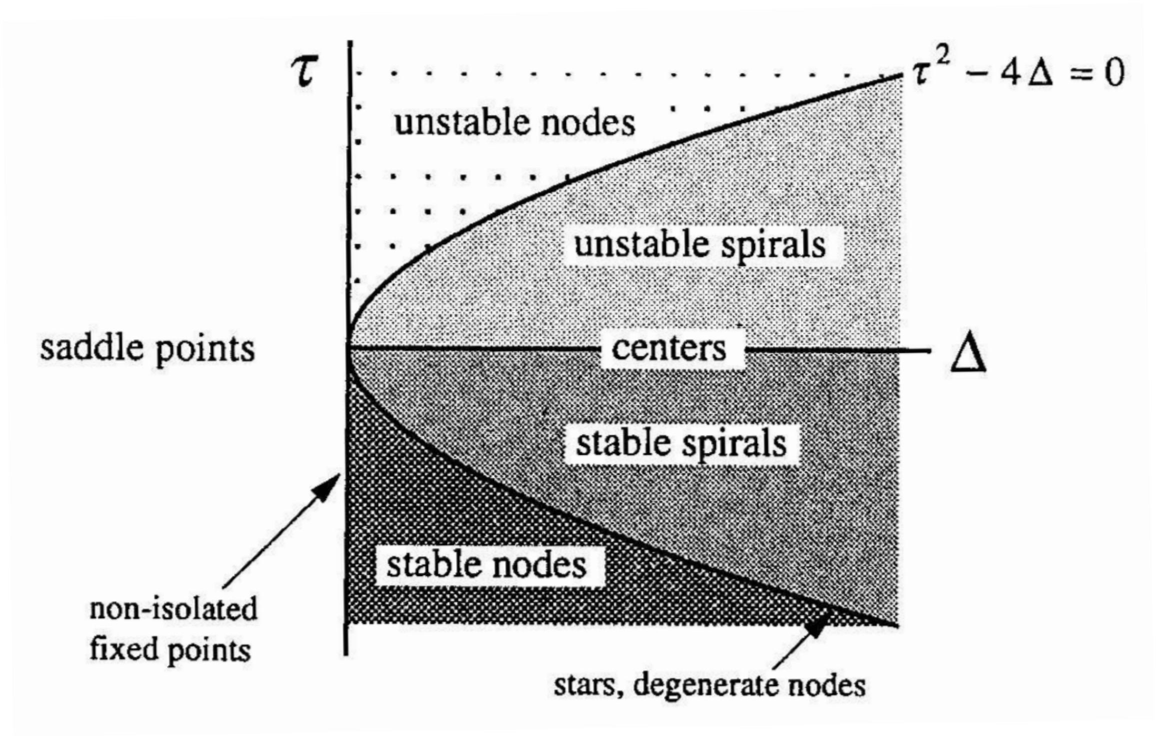
Proper and **improper nodes** arise when there is a repeated, nonzero eigenvalue, and this occurs precisely when

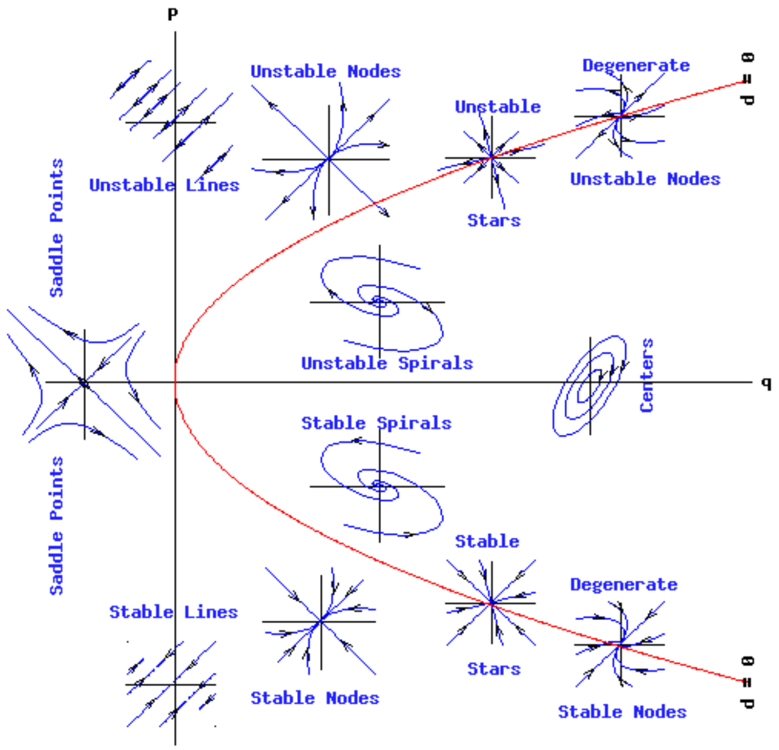
$$\tau \neq 0 \text{ and } \tau^2 - 4\Delta = 0 \qquad \Leftrightarrow \qquad 0 < \Delta = \frac{1}{4}\tau^2.$$

Spiral points arise when eigenvalues are complex $\alpha + i\beta$ with neither α nor β equal to 0; this occurs precisely when

$$\tau \neq 0 \text{ and } \tau^2 - 4\Delta < 0 \qquad \Leftrightarrow \qquad 0 < \frac{1}{4}\tau^2 < \Delta.$$

We gather all this information into the **stability diagram** below. Note that we are observing the $\Delta\tau$-plane here. An alternate version, one I found on the internet, which draws little characterization-diagrams for the various names, appears further down. It uses q and p for Δ and τ, respectively.


τ
$\tau^2 - 4\Delta = 0$
unstable nodes
unstable spirals
saddle points
centers
Δ
stable spirals
stable nodes
non-isolated
fixed points
stars, degenerate nodes


p
Unstable Nodes
Degenerate
d = 0
Unstable
Unstable Lines
Unstable Nodes
Stars
Saddle Points
Unstable Spirals
Centers
q
Stable Spirals
Saddle Points
Stable
Degenerate
Stable Lines
Stars
Stable Nodes
Stable Nodes
d = 0