Math 231, Thu 25-Mar-2021 -- Thu 25-Mar-2021
Differential Equations and Linear Algebra
Spring 2021

soln.

$$x_h(t) + x_p(t)$$
$$= \Phi(t)\vec{c} + \Phi\int \Phi^{-1} f \, dt$$

freedoms live

Thursday, March 25th 2021

Wk 8, Th
Topic:: System Wrap-up
Topic:: Converting to a 1st-order system

More nonhomogeneous systems
 - If x' = Ax + f(t) is made into an IVP, careful about when you determine c

 - Example: Exercise 5.5.12
   Use Cramer's rule

System approach to higher-order DEs
 - simple nth-order linear DE example
 - spring system

Ex.

$$\frac{d\vec{x}}{dt} = A\vec{x} + \sin(3t)\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Don't have A

Do have $\vec{f}(t) = \begin{bmatrix} 2\sin(3t) \\ 3\sin(3t) \end{bmatrix}$

$$\vec{x}_h(t) = \begin{bmatrix} \cos(3t) & \sin(3t) \\ 2\cos(3t) - 4\sin(3t) & 2\sin(3t) + 4\cos(3t) \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= \Phi(t) \cdot \vec{c}$$

$$\vec{x}_p(t) = \Phi(t)\int \underbrace{\Phi^{-1}(t)\,\vec{f}(t)}_{\vec{v}'(t)} \, dt$$

(is 2-2 vector fn.)

$$\vec{v}'(t) = \Phi^{-1}(t)\,\vec{f}(t) \qquad \text{Want } \vec{v}' = \begin{bmatrix} v_1' \\ v_2' \end{bmatrix}$$

Note: $\Phi(t)\,\vec{v}'(t) = \vec{f}(t)$

So I can use Cramer's Rule to get components of $\vec{v}'$

1st comp.

$$v_1' = \frac{\begin{vmatrix} 2\sin(3t) & \sin(3t) \\ 3\sin(3t) & 2\sin(3t)+4\cos(3t) \end{vmatrix}}{\begin{vmatrix} \cos(3t) & \sin(3t) \\ 2\cos(3t)-4\sin(3t) & 2\sin(3t)+4\cos(3t) \end{vmatrix}}$$

$$= \frac{2\sin(3t)\left[2\sin(3t)+4\cos(3t)\right] - 3\sin^2(3t)}{\cos(3t)\left[2\sin(3t)+4\cos(3t)\right] - \sin(3t)\left[2\cos(3t)-4\sin(3t)\right]}$$

$$v_2' = \frac{\begin{vmatrix} \cos(3t) & 2\sin(3t) \\ 2\cos(3t)-4\sin(3t) & 3\sin(3t) \end{vmatrix}}{\begin{vmatrix} \cos(3t) & \sin(3t) \\ 2\cos(3t)-4\sin(3t) & 2\sin(3t)+4\cos(3t) \end{vmatrix}}$$

Now would take v_1', v_2' and integrate them:

$$\vec{x}_p(t) = \Phi(t) \int \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} dt$$

$$= \Phi(t) \begin{bmatrix} \int v_1' \, dt \\ \int v_2' \, dt \end{bmatrix}$$

Integrals don't look appetizing

1^{st}-order systems is a gateway to solving many problems that, on first look, don't appear similar.

Ex.) 3^{rd}-order DE

$$y''' + 2y' - 7y = \cos t$$

(dep. var. y, ind. var. t)

Can introduce new dep. vars.

Set $x_1(t) = y(t)$ (rename y)

$x_2(t) = y'(t)$ (rename y')

$x_3(t) = y''(t)$ (rename y'')

Stop one deriv. shy of highest deriv. of y seen

Note: $x_1' = \frac{d}{dt} y = y' = x_2$

$x_2' = \frac{d}{dt} y' = y'' = x_3$

$\cancel{x_3' = \frac{d}{dt} y''}$ ~~=~~ ~~call~~ $\cancel{x_4}$ Don't do

Instead, use new names in original DE

$$y''' + 2y' - 7y = \cos t$$

Under new names

$$x_3' + 2x_2 - 7x_1 = \cos t$$

arrange so x_3' is alone

$$x_3' = -2x_2 + 7x_1 + \cos t$$

coupled w/

$$x_1' = x_2$$

$$x_2' = x_3$$

We have 1^{st}-order system

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ -2x_2 + 7x_1 + \cos t \end{bmatrix}$$

$$= x_1 \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cos t \end{bmatrix}$$

$$\vec{x}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 7 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cos t \end{bmatrix}$$

$$A \quad \vec{x} \quad + \quad \vec{f}(t)$$

like a Ch. 3 problem

Invent new dep. vars.

Let $x_1 = u_1$ $\qquad x_1' = x_2$

$x_2 = u_1'$ $\qquad x_2' =$ ______

$x_3 = u_2$ $\qquad x_3' = x_4$

$x_4 = u_2'$ $\qquad x_4' =$ ______