Math 231, Fri 26-Mar-2021 -- Fri 26-Mar-2021
Differential Equations and Linear Algebra
Spring 2021

Friday, March 26th 2021

Wk 8, Fr
Topic:: Higher order linear DEs
Read:: ODELA 4.1

Example:
 y'' + 4y' - 5y = 0 — linear, 2^{nd}-order, homogeneous, constant coeff.
 describe
 should expect:
   solution has 2 degrees of freedom, expressed as constants
   if conditions were added to nail down constants, would need 2 constraints

2 ICs required in order to hope for uniqueness of solution.

 convert to a 1st order system and solve

 use solution to find a streamlined method

Other examples of linear homogeneous, constant-coefficient DEs:

$$y'' + 4y' - 5y = 0$$

Define $x_1 = y$, $x_2 = y'$ } "New" dep. vars. x_1, x_2

By their definitions,

$$\boxed{\begin{aligned} x_1' &= x_2 \\ x_2' &= -4x_2 + 5x_1 = 0 \end{aligned}} \quad (\star)$$

Rewrite

Solved for x_2'

$$x_2' + 4x_2 - 5x_1 = 0$$

($\star$) in matrix form

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ -4 & 5 \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ -4 \end{bmatrix} + x_1 \begin{bmatrix} 0 \\ 5 \end{bmatrix}}$$

True, but don't do

Undesirable because it's not

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

Our vector is in wrong order

Fixed $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 5 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ Good!

$$\frac{d\vec{x}}{dt} = A \quad \vec{x}$$

Find eigenpairs (eigenvalue along w/ corresp. basis e-vecs) for A.

λ	corresp. basis e-vec
-5	$\begin{bmatrix} 1 \\ -5 \end{bmatrix}$
1	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

General soln.

$$\vec{x}(t) = c_1 e^{-5t} \begin{bmatrix} 1 \\ -5 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{-5t} & e^{t} \\ -5e^{-5t} & e^{t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Originally $y'' + 4y' - 5y = 0$, and

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{-5t} + c_2 e^{t} \\ -5c_1 e^{-5} + c_2 e^{t} \end{bmatrix} = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}$$

So, the original DE has general soln.

$$y(t) = c_1 e^{-5t} + c_2 e^{t}$$

Now that we're solved, observe that it is a linear comb. of exponential fns.

Idea: Assume fns. of form $y = e^{\lambda t}$ solve our DE and see where it leads.

Now start on problem: $y'' + 4y' - 5y = 0$

Assume (look for) solns. of form $y = e^{\lambda t}$.

Entails inserting

$e^{\lambda t}$ for y

$\lambda e^{\lambda t}$ for y'

$\lambda^2 e^{\lambda t}$ for y''

Original becomes

$$\lambda^2 e^{\lambda t} + 4\lambda e^{\lambda t} - 5e^{\lambda t} = 0$$

$$(\lambda^2 + 4\lambda - 5)e^{\lambda t} = 0$$

or

$$\lambda^2 + 4\lambda - 5 = 0$$

$$(\lambda + 5)(\lambda - 1) = 0$$

Same polynomial eqn. as $\det(A - \lambda I) = 0$

So, continue to call it the characteristic eqn.

Yields the roots $\lambda = -5, 1$ (same as e-vals of A)

Both e^{-5t} and e^{t} solve our DE

and, being homogeneous, all linear combs. do as well

called the "superposition principle"

Thus $y(t) = c_1 e^{-5t} + c_2 e^{t}$ (same result both ways)

2^{nd} problem (only using 2^{nd} approach)

$$3y'' - 4y' + y = 0, \quad \left\{ \begin{array}{l} y(0) = 3 \\ y'(0) = 1 \end{array} \right\}$$ 2 initial conditions

Assume solns. of form $y(t) = e^{\lambda t}$

$$3\lambda^2 e^{\lambda t} - 4\lambda e^{\lambda t} + e^{\lambda t} = 0 \quad \text{or} \quad 3\lambda^2 - 4\lambda + 1 = 0$$

$$(3\lambda - 1)(\lambda - 1) = 0$$

$\Rightarrow \lambda = \frac{1}{3}, 1$ are the characteristic values

general soln.

$$y(t) = c_1 e^{t/3} + c_2 e^{t}$$

Differentiating $y'(t) = \frac{1}{3} c_1 e^{t/3} + c_2 e^{t}$

ICs require

at $t=0$

Use at $t=0$

$$3 = y(0) = c_1 e^0 + c_2 e^0 = c_1 + c_2$$

$$1 = y'(0) = \frac{1}{3} c_1 e^0 + c_2 e^0 = \frac{1}{3} c_1 + c_2$$

As matrix problem

$$\begin{bmatrix} 1 & 1 \\ \frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Cramer's Rule

$$c_1 = \frac{\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1/3 & 1 \end{vmatrix}} = \frac{3-1}{1 - 1/3} = \frac{2}{2/3} = 3$$

$$c_2 = \frac{\begin{vmatrix} 1 & 3 \\ 1/3 & 1 \end{vmatrix}}{\text{same denom.}} = \frac{1-1}{2/3} = 0$$

Use c_1, c_2 in general soln. — ones which enforce ICs to hold:

Soln. of my IVP has soln. $\boxed{y(t) = 3e^{t/3}.}$