Math 231, Mon 29-Mar-2021 -- Mon 29-Mar-2021
Differential Equations and Linear Algebra
Spring 2021

Monday, March 29th 2021

Wk 9, Mo
Topic:: Higher order linear DEs intro
Read:: ODELA 4.2

Exam 2 coming next Monday
 - corresponds to material from Chapters 2 and 3

Friday is Good Friday
 - it is on the calendar as a class day
 - I intend to
    make it a day for Q&A only
    make it a "virtual" day, conducted through Teams
    make attendance optional

Homogeneous, constant-coefficient <u>nth-order DEs</u> (linear)
 - method is not for non-homogeneous, and
   cannot be expected to work if coefficients are non-constant
 - How to deal with
    nonreal characteristic values
    repeated characteristic values
 - Examples
    $y'' + 3y' + 2y = 0$, $y(2)=1$, $y'(2)=-1$
    $y^{(4)} - y''' - 13y' + y' + 12y = 0$,
       subj. to $y(0) = 2$, $y'(0)=5$, $y''(0)=-29$, $y'''(0)=-7$.

       soln is $-2e^{-3t} + e^{-t} + 4e^t - c^{4t}$

    $y'' + 4y' + 4y = 0$
    $y'' + 2y' + 10y = 0$

Ex.

$$y'' + 3y' + 2y = 0$$

2nd-order, homogeneous linear, const. coeffs.

Method: assume exponential solns

$$y = e^{\lambda t} \qquad \left(\text{so } y' = \lambda e^{\lambda t},\ y'' = \lambda^2 e^{\lambda t}\right)$$

See what choices of λ make this work

Get, after inserting our proposed soln. form, a characteristic eqn

$$\lambda^2 + 3\lambda + 2 = 0$$

$$(\lambda+2)(\lambda+1) = 0 \implies \lambda = -1, -2$$

characteristic values

e^{-t}, e^{-2t} both solve

By superposition, so does any linear comb (general soln.)

$$y(t) = c_1 e^{-t} + c_2 e^{-2t}$$

If on top of DE we have ICs: $\begin{cases} y(2) = 1 \\ y'(2) = -1 \end{cases}$

To satisfy the ICs, we'll need a formula for y'

$$y'(t) = -c_1 e^{-t} - 2c_2 e^{-2t}$$

Now insist that

$$\boxed{\begin{aligned} y(2) &= c_1 e^{-2} + c_2 e^{-4} = 1 \\ y'(2) &= -c_1 e^{-2} - 2c_2^{-4} = -1 \end{aligned}}$$

2 eqns., 2 unknowns c_1, c_2

In matrix form

$$\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Ex.)

$$y^{(4)} - y''' - 13y'' + y' + 12y = 0 \qquad \text{linear, homog., const. coeff.}$$

Our char. eqn.

$$\lambda^4 - \lambda^3 - 13\lambda^2 + \lambda + 12 = 0$$

A graph shows zeros of $y = x^4 - x^3 - 13x^2 + x - 12$ at $-3, -1, 1, 4$

So, $(\lambda+3)(\lambda+1)(\lambda-1)(\lambda-4) = 0$

but more importantly,

$$e^{-3t},\ e^{-t},\ e^{t},\ e^{4t}$$

all solve our DE, and so does any linear comb:

$$c_1 e^{-3t} + c_2 e^{-t} + c_3 e^{t} + c_4 e^{4t}$$

Q: What if my char. poly. has nonreal roots?

Recall, if a 1^{st}-order system has a nonreal root

$\lambda = \alpha + \beta i$ w/ corresp e-vector $\vec{v} = \vec{u} + i\vec{w}$

that

1. Any rescaling of $\vec{v}$ is also an e-vector
   - so, in particular we can rescale $\vec{v}$ so that its first component is 1.

$$\begin{bmatrix} 1 \\ * \\ \vdots \\ * \\ * \end{bmatrix} = \vec{v} = \underbrace{\begin{bmatrix} 1 \\ * \\ \vdots \\ * \end{bmatrix}}_{\vec{u}} + i \underbrace{\begin{bmatrix} 0 \\ * \\ \vdots \\ * \end{bmatrix}}_{\vec{w}}$$

complex nos.

2. $\lambda = \alpha - \beta i$, w/ corresp. e-vector $\vec{u} - i\vec{w}$, serve as an e-pair

How used in Ch. 3,

replaced system solns, $e^{(\alpha+\beta i)t}(\vec{u} + i\vec{w})$, $e^{(\alpha-\beta i)t}(\vec{u} - i\vec{w})$

by $e^{\alpha t}\left[\cos(\beta t)\vec{u} - \sin(\beta t)\vec{w}\right]$, $e^{\alpha t}\left[\sin(\beta t)\vec{u} + \cos(\beta t)\vec{w}\right]$

Back to our problem:

Though we may not transform our nth-order, linear, homog., const. coeff problem to a system, we know that if we did, the soln. $\vec{x}(t)$

$$\vec{x}(t) = \begin{bmatrix} y(t) \\ y'(t) \\ y''(t) \\ \vdots \\ y^{(n-1)}(t) \end{bmatrix}$$

Note $\cos(\beta t)\vec{u} - \sin(\beta t)\vec{w} = \left(\cos(\beta t)\begin{bmatrix} 1 \\ * \\ * \\ \vdots \\ * \end{bmatrix} - \sin(\beta t)\begin{bmatrix} 0 \\ * \\ \vdots \\ * \end{bmatrix}\right) e^{\alpha t}$

$$= \begin{bmatrix} \cos(\beta t) \\ \text{other things} \\ \vdots \end{bmatrix} \cdot e^{\alpha t}$$

and

$$e^{\alpha t}\left(\sin(\beta t)\,\vec{u} + \cos(\beta t)\,\vec{v}\right) = \quad \cdots \quad = e^{\alpha t} \begin{bmatrix} \sin(\beta t) \\ \text{other coords} \\ \vdots \end{bmatrix}$$

Upshot: $e^{\alpha t}\cos(\beta t)$, $e^{\alpha t}\sin(\beta t)$ can serve as our two solns. contributed to the general soln. by char. vals. $\alpha \pm \beta i$.

Ex.) $y'' + 2y' + 10y = 0$

$$\lambda^2 + 2\lambda + 10 = 0$$

$$(\lambda^2 + 2\lambda + 1) + 9 = 0$$

$$(\lambda + 1)^2 = -9$$

$$\lambda + 1 = \pm\sqrt{-9} = \pm 3i$$

$$\lambda = -1 \pm 3i \qquad \text{Here } \alpha = -1,\ \beta = 3$$

By our work above, both

$$e^{-t}\cos(3t), \quad e^{-t}\sin(3t)$$

solve our DE. <u>And</u>, neither has $i = \sqrt{-1}$ in them!

So, also applying superposition, linear combs. also solve

$$y(t) = c_1 e^{-t} \cos(3t) + c_2 e^{-t} \sin(3t)$$