Math 231, Wed 31-Mar-2021 -- Wed 31-Mar-2021
Differential Equations and Linear Algebra
Spring 2021

-------------------------------
Wednesday, March 31st 2021
-------------------------------

Wk 9, We
Topic:: Linear homogeneous DEs
Read:: ODELA [redacted] 4.1

open up with practice:

$4y'' + 4y' + 3y = 0$

$4y''' + 12y'' + 9y' + 27y = 0$

$y'' + 4y' + 4y = 0$

has repeated root

turn to system to see how to proceed

observations:

- eigenvalue is degenerate
  - always true for nth-order DE converted to system
  - in fact: it will always have GM = 1 regardless of how large AM
- can always make
  - 1st coordinate of eigenvector 1
  - 1st coordinate of generalized eigenvector 0
- upshot: to usual $e^{-2t}$ can also use $te^{-2t}$

$y''' + 3y'' + 3y' + y = 0$

Ex.] $4y'' + 4y' + 3y = 0$

$4\lambda^2 + 4\lambda + 3 = 0$

$\lambda = \frac{1}{8}\left(-4 \pm \sqrt{16 - (4)(4)(3)}\right)$

$= -\frac{1}{2} \pm \frac{1}{8}\sqrt{-32}$

$= -\frac{1}{2} \pm \frac{1}{8} i \sqrt{16} \cdot \sqrt{2}$

$= -\frac{1}{2} \pm \frac{1}{2} i \sqrt{2}$

$= \alpha \pm \beta i$ w/

$\alpha = -1/2$

$\beta = \sqrt{2}/2$

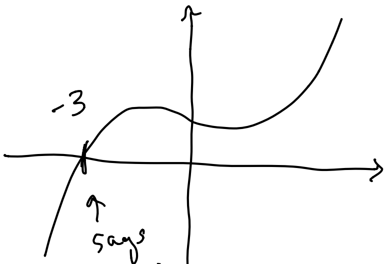
Continuing

Yesterday: If nonreal, conjugate characteristic values $\alpha \pm \beta i$, then $e^{\alpha t}\cos(\beta t)$, $e^{\alpha t}\sin(\beta t)$ are real (no $\sqrt{-1}$) solutions, and so is any linear comb

$4y'' + 4y' + 3y = 0$ has general soln.

$$y(t) = c_1 e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{2}}{2} t\right) + c_2 e^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{2}}{2} t\right)$$

Ex.] $4y''' + 12y'' + 9y' + 27y = 0$



says $(\lambda+3)$ is a factor

Char. eqn. $4\lambda^3 + 12\lambda^2 + 9\lambda + 27 = 0$

$$(4\lambda^3 + 12\lambda^2) + (9\lambda + 27) = 0$$

$$\lambda+3 \overline{\big) 4\lambda^3 + 12\lambda^2 + 9\lambda + 27}^{\;4\lambda^2 + 9}$$

$$4\lambda^2(\lambda + 3) + 9(\lambda + 3) = 0$$

$$(4\lambda^2 + 9)(\lambda + 3) = 0$$

So either $\lambda = -3$ or $4\lambda^2 + 9 = 0$

$$4\lambda^2 = -9$$

$$\lambda^2 = -9/4$$

$$\lambda = \pm\sqrt{-9/4} = \pm \frac{3}{2} i$$

3 characteristic values $\lambda = \boxed{-3}, \underbrace{\frac{3}{2} i, -\frac{3}{2} i}$ $\left(\begin{matrix}\alpha = 0 \\ \beta = 3/2\end{matrix}\right)$

corresp. solns. $e^{-3t}, \cos\left(\frac{3}{2}t\right), \sin\left(\frac{3}{2}t\right)$

general soln. is linear comb:

$$y(t) = c_1 e^{-3t} + c_2 \cos\left(\frac{3}{2}t\right) + c_3 \sin\left(\frac{3}{2}t\right).$$

Ex.] $y'' + 4y' + 4y = 0$ $\longrightarrow$ char eqn $\lambda^2 + 4\lambda + 4 = 0$

$(\lambda + 2)^2 = 0$

$\Longrightarrow$ one (repeated) char. val. $\lambda = -2$.

So $e^{-2t}$ solves

But, it's a 2nd-order DE, and we expect a 2nd L.I. soln. to join w/ this one in linear comb.

Q: Where / what is a 2nd ?

---

To find, take the system-conversion path.

Invent new dep. vars.

Let $\left.\begin{aligned} x_1 &= y \\ x_2 &= y' \end{aligned}\right\}$ Get ① $\frac{dx_1}{dt} = y' = x_2$

For 2nd DE in our 1st-order system, rewrite $y'' + 4y' + 4y = 0$

$$y'' + 4y' + 4y = 0$$

Inserting new names

$$\frac{dx_2}{dt} + 4x_2 + 4x_1 = 0$$

and solve

$$② \quad \frac{dx_2}{dt} = -4x_1 - 4x_2$$

① and ② together form a 1st-order system

$x_1' = x_2$

$x_2' = -4x_1 - 4x_2$

or $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

or $\vec{x}' = A\vec{x}$, a homog., const. coeff., 1st-order system

(Ch. 3)

eigenpairs of $A$:

| $\lambda$ | basis of evecs |
|---|---|
| $-2$<br>degenerate | $\begin{bmatrix}1\\-2\end{bmatrix}$ |

Need two L.I. solns. here, too

$e^{-2t}\begin{bmatrix}1\\-2\end{bmatrix}$ is our 1st

2nd? It takes form $e^{-2t}\left(\vec{u} + t\begin{bmatrix}1\\-2\end{bmatrix}\right)$

with $\vec{u}$ solves $\left[A - (-2)I\right]\vec{u} = \begin{bmatrix}1\\-2\end{bmatrix}$

$$\left[\underbrace{\begin{array}{cc}2 & 1\\ -4 & -2\end{array}}_{A+2I}\middle|\begin{array}{c}1\\-2\end{array}\right] \longleftrightarrow \left[\begin{array}{cc|c}2 & 1 & 1\\ 0 & 0 & 0\end{array}\right]$$

says components in $\vec{u} = \langle u_1, u_2\rangle$ satisfy $2u_1 + u_2 = 1$

A candidate $\vec{u}$, the one w/ $u_1 = 0$, is $\vec{u} = \begin{bmatrix}0\\1\end{bmatrix}$

2nd soln.

$$e^{-2t}\left(\begin{bmatrix}0\\1\end{bmatrix} + t\begin{bmatrix}1\\-2\end{bmatrix}\right) = e^{-2t}\begin{bmatrix}t\\1-2t\end{bmatrix}$$

So system has general solution

$$\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \vec{x}(t) = c_1 e^{-2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} t \\ 1-2t \end{bmatrix}$$

So isolating our view to the 1st component

$$y(t) = c_1 e^{-2t} + c_2 t e^{-2t}$$

Upshot: When you have a repeated char. value

- repeated twice $\lambda = \lambda_1$

  $e^{\lambda_1 t}, \quad t e^{\lambda_1 t}$ serve as L.I. solns.

- repeated 3 times $\lambda = \lambda_1$

  $e^{\lambda_1 t}, \quad t e^{\lambda_1 t}, \quad t^2 e^{\lambda_1 t}$