Classifying equilibrium at the origin

Context

- dealing homogeneous <u>planar</u> (2x2 matrix) linear system w/ constant coeffs

$$\vec{x}\,' = A_{2\times 2}\,\vec{x}$$

- Note: the constant vector fn. $\vec{x}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ solves. Since it stays constant, it's an equilibrium soln.
- Any other vector in Null(A) would also be an equilibrium.

Can have: (Stability)

1. All solns. running toward $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, "reaching it" in limit as $t \to \infty$

   (<u>global asymptotic stability</u>)

2. Solns. may vary in distance from $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ at different t-vals., but they neither go to it as $t \to \infty$ nor do they stray infinitely far from it as $t \to \infty$.

   (like children playing in a big(?) yard

   Classify $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as <u>stable</u>.

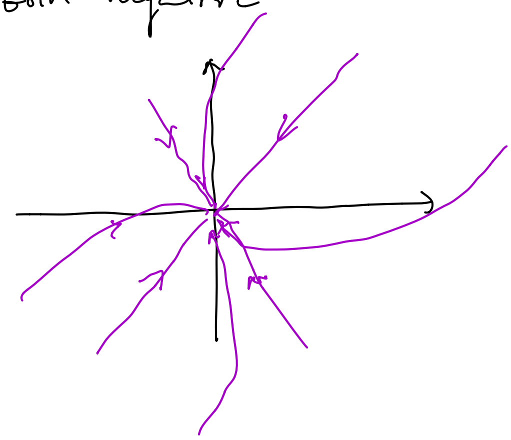
3. Even solns. w/ Initial points arbitrarily close to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ can go "infinitely far" from $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as $t \to \infty$.

Classify $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as unstable.

It comes down to eigenvalues

A. both e-vals are real

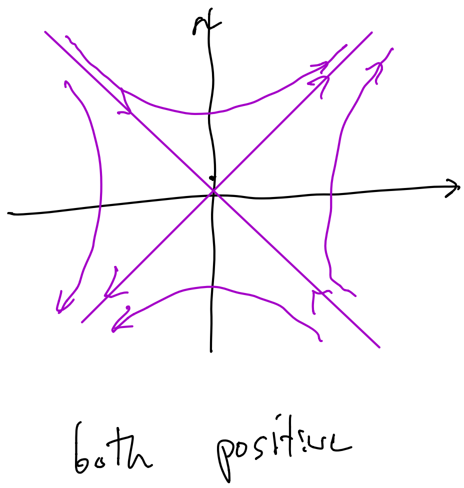
A1. both negative



global asymptotic stability

node

A2. one is neg., one pos.



unstable saddle point

A3. both positive

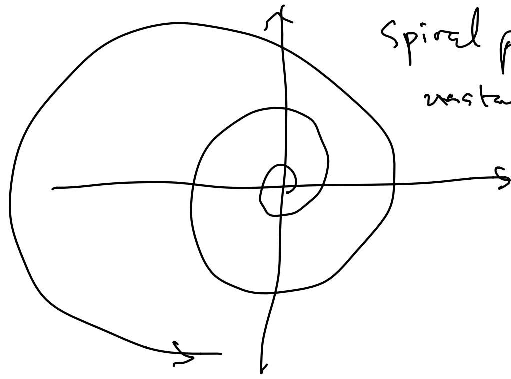
unstable

node

$e^{\alpha t}\left[\vec{v}\cos(\beta t) - \vec{w}\sin(\beta t)\right]$

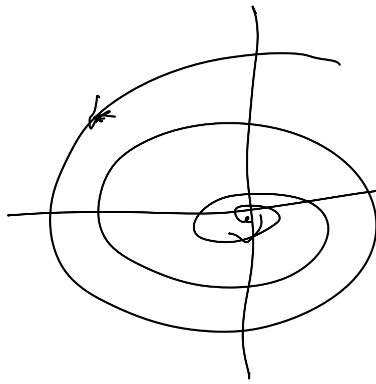
B. both e-vals are nonreal $\alpha \pm \beta i$

B1. $\alpha > 0$



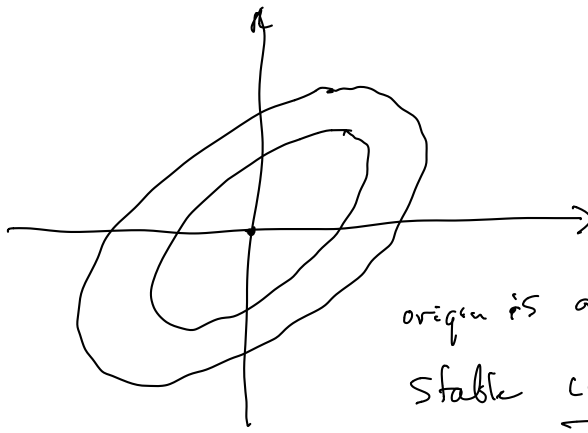
spiral point
unstable

B2. $\alpha < 0$



globally asymp. stable
spiral point

B3. $\underline{\alpha = 0}$



origin is a
stable <u>center</u>

Converting a DE (or a system of DEs) to a 1^{st}-order system

Ex.)

ICs

$y_1(0) = 4$

$y_2(0) = 1$

$y_2'(0) = -1$

$$y_1' + 2y_2 - \cos t = 0$$

$$2y_1 + 3y_2 - y_2' + 3y_2'' = t^2$$

system

$y_1' = \cos t - 2y_2$

vars: independent? t

dependent? y_1, y_2

$y_2'' = \left(t^2 - 2y_1 - 3y_2 + y_2'\right)/3$

$x_3' = \left(t^2 - 2x_1 - 3x_2 + x_3\right)/3$

To find an equivalent 1^{st}-order system

New vars

$$x_1 = y_1$$

$$x_2 = y_2$$

$$x_3 = y_2'$$

$\Longrightarrow x_2' = y_2' = x_3$

Want (normal form) $\vec{x}' = \vec{f}(\quad)$ only t, x_1, x_2, x_3 appear, but not derivs.

$$x_1' = \cos t - 2x_2$$

$$x_2' = x_3$$

$$x_3' = \left(t^2 - 2x_1 - 3x_2 + x_3\right)/3$$

In matrix form

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 + \cos t \\ x_3 \\ -\frac{2}{3}x_1 - x_2 + \frac{1}{3}x_3 + \frac{1}{3}t^2 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} 0 \\ 0 \\ -2/3 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ 1/3 \end{bmatrix} + \begin{bmatrix} \cos t \\ 0 \\ \frac{1}{3}t^2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ -2/3 & -1 & 1/3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \cos t \\ 0 \\ \frac{1}{3}t^2 \end{bmatrix}$$

$$= A\vec{x} + \vec{f}(t).$$

Rewriting ICs

$$\underline{\vec{x}(0)} = \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}$$

For 1^{st} order DE: $y' = a(t)y + f(t)$

$$\varphi(t) = e^{\int a(t)\,dt} \xrightarrow{\text{basis fn for}} x_h(t) = c\,\varphi(t)$$

Var. of params formula: $x_p(t) = \varphi(t) \int \frac{f(t)}{\varphi(t)}\,dt$

general soln.: $x(t) = x_h(t) + x_p(t)$

For 1^{st}-order system $\vec{x}' = A\vec{x} + \vec{f}(t)$

Fnd'l matrix $\Phi(t)$, cols. built from L.I. solns. to $\vec{x}' = A\vec{x}$.

Find $\vec{x}_p(t) = \Phi(t) \int \Phi^{-1}(t)\,\vec{f}(t)\,dt$

gen'l soln

$$\underbrace{\Phi(t)\,\vec{c}}_{\vec{x}_h(t)} + \vec{x}_p(t)$$