Trig identity: $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$$\Rightarrow \quad R\cos(\omega t - \delta) = \underbrace{R\cos\delta}_{c_1}\cos(\omega t) + \underbrace{R\sin\delta}_{c_2}\sin(\omega t)$$

Ex] Solve $y'' + 4y = 0$, subject to $y(0) = 3$, $y'(0) = -8$.

Seek solns. of form $y = e^{\lambda t}$ $\quad (\Rightarrow y' = \lambda e^{\lambda t},\ y'' = \lambda^2 e^{\lambda t})$

Characteristic eqn. $\lambda^2 + 4 = 0$

$\lambda^2 = -4$

$\lambda = \pm 2i = 0 \pm 2i$ (nonreal char. vals)

last week: If $\lambda = \alpha \pm \beta i$ are roots of char. eqn., then

$\left.\begin{array}{l} \cancel{e^{\alpha t}\cdot e^{i\beta t}} \\ \cancel{e^{\alpha t}\cdot e^{i(-\beta t)}} \end{array}\right\}$ ~~both solns.~~ replace with $e^{\alpha t}\cos(\beta t)$, $e^{\alpha t}\sin(\beta t)$

$e^{0t}\cdot\cos(2t)$, $e^{0t}\sin(2t)$ both solve

as well as as every linear combination, too

$y(t) = c_1\cos(2t) + c_2\sin(2t)$ (general soln)

$\Rightarrow y'(t) = -2c_1\sin(2t) + 2c_2\cos(2t)$

Now apply ICs

$3 = y(0) = c_1$

$-8 = y'(0) = 2c_2$ $\quad \Rightarrow \quad c_1 = 3,\ c_2 = -4$

Unique soln. of the IVP: $y(t) = 3\cos(2t) - 4\sin(2t)$ acceptable answer

possible to rewrite as $R\cos(\omega t - \delta)$

Let $c_1 = R\cos\delta$, $c_2 = R\sin\delta$ then reverse of this relationship

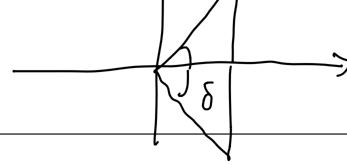
$R^2 = c_1^2 + c_2^2$ or $R = \sqrt{c_1^2 + c_2^2}$

and $\cos\delta = \frac{c_1}{R}$, $\sin\delta = \frac{c_2}{R}$

In our problem $R = \sqrt{3^2 + (-4)^2} = 5$

$\sin\delta = -4/5$, $\cos\delta = 3/5$ $\Rightarrow \delta$ in Quadrant IV

$\delta = \cos^{-1}(3/5) =$ gives Quadrant I angle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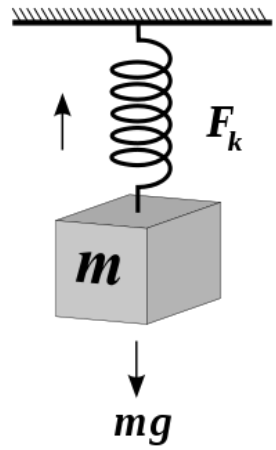
2nd Order DE Models

We will look at two main models that are not only 2nd order, but have constant coefficients. They are the spring–mass assembly (which gets most of our time) and simple electric circuits.

Spring–mass assembly



Consider a mass suspended at the end of a spring, as shown at right. If the natural length of the spring—i.e., without any *load*—is ℓ, and our object with mass m stretches the spring to a new length $(\ell + L)$ then, in the absence of motion, we have two forces in equilibrium: $mg = F_k$. Here F_k represents the restorative force of the spring, which has been established experimentally under small displacements to have expression $F_k = kL$, where $k > 0$ is an internal spring constant.

When this same assembly is in motion, we let $u(t)$ denote the difference in the actual length of the spring and the equilibrium length $(\ell + L)$. [Note: We will take the downward direction as positive, so $u > 0$ when the string is stretched longer than $(\ell + L)$.] Newton's 2nd Law now says

$$m\frac{d^2u}{dt^2} \;=\; mg + F_k + F_r + F_e \;=\; mg - k(L+u) + F_r + F_e \;=\; -ku + F_r + F_e,$$

where F_r represents damping force, and F_e encapsulates any external forces that drive the motion. In a number of applications, F_r is reasonably approximated as being proportional to speed (and opposite in direction to it), which gives $F_r = -\gamma\frac{du}{dt}$, where $\gamma > 0$ is the constant of proportionality. Thus, we have the 2nd order DE model for motion in this spring–mass assembly

means $\gamma = 0$ means $F_e = 0$

$$\boxed{m\frac{d^2u}{dt^2} + \gamma\frac{du}{dt} + ku \;=\; F_e.} \qquad (1)$$

m = mass
γ = damping constant
k = spring constant

Undamped, unforced vibrations

We assume, for now, that our spring–mass assembly experiences no damping (so $F_r = 0$). It is interesting to see the implications of this, even if no such spring exists. If we assume, also, there are no external forces (so $F_e = 0$) in (1), we get the linear homogeneous DE

$mu'' + ku = 0 \Rightarrow u'' + \frac{k}{m}u = 0$

$$\frac{d^2u}{dt^2} + \omega_0^2 u \;=\; 0,$$

$y'' + 4y = 0$ special case of undamped, unforced spring

with $\omega_0 = \sqrt{k/m}$. (You may recall this same DE arose from *linearizing* the pendulum equation—i.e., by assuming $\sin\theta \approx \theta$.) The general solution of this problem is

$$u(t) \;=\; c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t),$$

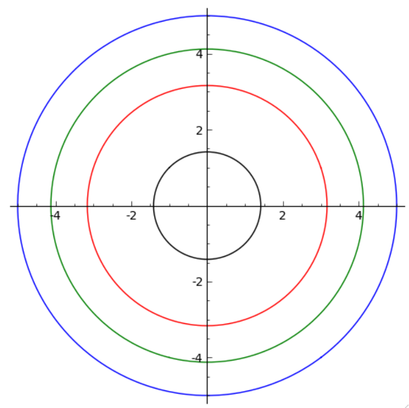
ω_0 is called angular frequency
$\frac{\omega_0}{2\pi}$ " " frequency

$\frac{2\pi}{\omega_0}$ is the period

where the role of ω_0, called the **natural frequency**, is evident. The resulting periodic displacement u describes what is called **simple harmonic motion**.

One may plot the solution (for various choices of c_1, c_2) in the tu-plane, of course. A less-familiar way to plot solutions is parametrically in the uu'-plane (called the **phase plane**), and we do so below (using SAGE) for each combination of choices $c_1 = -3, 1$ and $c_2 = -1, 4$, with $\omega_0 = 1$ (fixed).

```
var('t')
u1(t) = -3*cos(t) - sin(t)
u2(t) = -3*cos(t) + 4*sin(t)
u3(t) = cos(t) - sin(t)
u4(t) = cos(t) + 4*sin(t)
p1 = parametric_plot((u1(t), diff(u1,t)), (t,-pi,pi), color='red')
p2 = parametric_plot((u2(t), diff(u2,t)), (t,-pi,pi), color='blue')
p3 = parametric_plot((u3(t), diff(u3,t)), (t,-pi,pi), color='black')
p4 = parametric_plot((u4(t), diff(u4,t)), (t,-pi,pi), color='green')
pall = p1 + p2 + p3 + p4
pall.show()
```



Forced, undamped vibrations

While external forces are not always periodic, we experience them enough (the motions of a child's legs to drive a swing on a playground, regular imperfections in the highway, etc.) to make them worth a look. So, we take $F_e = F_0 \cos(\omega t)$ with $F_0 > 0$. When $\omega \neq \omega_0$, we know from homework that the general solution of

$$\frac{d^2u}{dt^2} + \omega_0^2 u = \frac{F_0}{m}\cos(\omega t) \tag{2}$$

is

$$u(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos(\omega t).$$

When the frequency of the forcing term ω is a value close to the natural frequency ω_0, the amplitude $F_0/(m|\omega_0^2 - \omega^2|)$ of the final term can be very large. Still, none of the terms have an amplitude that *grows* over time, so if the peak value which results from their combination is not so large that it breaks the spring, vibrations continue indefinitely.

It is interesting to consider problem (2) along with ICs $u(0) = 0$, $u'(0) = 0$ that place the spring–mass assembly initially at rest. It may be shown, in this case (the details are given in our text on p. 212 where, along with other, more familiar steps, the trigonometric identity

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha - \beta}{2}\right)\sin\left(\frac{\alpha + \beta}{2}\right)$$

is used), that the solution is

$$u(t) = \left[\frac{2F_0}{m(\omega_0^2 - \omega^2)}\sin\left(\frac{(\omega_0 - \omega)t}{2}\right)\right]\sin\left(\frac{(\omega_0 + \omega)t}{2}\right).$$

When ω is similar in size to ω_0, this may be viewed as a rapidly oscillating function (frequency is $(\omega_0 + \omega)/2$) with a slowly varying amplitude (the part in the square brackets, having frequency $(\omega_0 - \omega)/2$). This is phenomenon is known variously as **beats** or **amplitude modulation**. Though we have derived it under the impossible scenario of *no damping*, something very much like it occurs physically when one tuning fork is used to excite another tuning fork with a slightly different natural frequency, resulting in a note that seems to get louder and softer periodically.

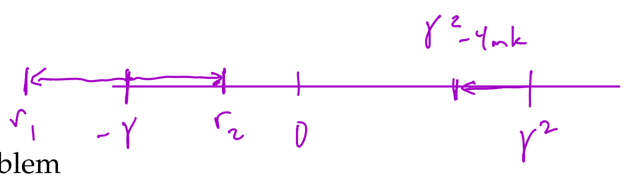
We have a very different situation when, in (2), the frequency ω of the forcing term is equal to the natural frequency ω_0. In this case, the solution of (2) (again a result from homework) is

$$u(t) \;=\; c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{F_0}{2m\omega_0}\, t\sin(\omega_0 t).$$

The amplitude $F_0 t/|2m\omega_0|$ in the final term grows with time, so that no spring can take the load indefinitely. This is what we call **resonance**. As we shall see, this phenomenon cannot occur in the presence of damping, making resonance (i.e., an amplitude that grows without bound) purely a mental construct.

Damped vibrations , unforced

$\gamma > 0$ $(F_e = 0)$



We first consider the homogeneous (unforced) problem

$$mu'' + \gamma u' + ku \;=\; 0. \tag{3}$$

Since the coefficients m, γ and k are all positive, the roots of our characteristic equation

$$mx^2 + \gamma x + k \;=\; 0, \qquad \text{given by} \qquad r_{1,2} = \frac{1}{2m}\left(-\gamma \pm \sqrt{\gamma^2 - 4mk}\right),$$

$\gamma^2 - 4mk$

are $\quad m\lambda^2 + \gamma\lambda + k = 0$

- $\gamma^2 - 4mk > 0$ real, with $r_1 \neq r_2$ and both $r_{1,2} < 0$, yielding general solution
$$u_h(t) \;=\; c_1 e^{r_1 t} + c_2 e^{r_2 t}. \longrightarrow 0 \text{ as } t \to \infty$$
This case, occuring when $\gamma > 2\sqrt{mk}$, is called **overdamping**.
- $\gamma^2 - 4mk = 0$ real, with $r_1 = r_2 = -\gamma/(2m)$, yielding general solution
$$u_h(t) \;=\; c_1 e^{-\gamma t} + c_2 t e^{-\gamma t}. \longrightarrow 0 \text{ as } t \to \infty$$
This case, occurring when $\gamma = 2\sqrt{mk}$, is called **critical damping**.
- $\gamma^2 - 4mk < 0$ nonreal $r_{1,2} = \alpha \pm i\beta$, with $\alpha = -\gamma/(2m)$ and $\beta = \sqrt{4mk - \gamma^2}/(2m)$, yielding general solution
$$u_h(t) \;=\; c_1 e^{-\gamma t/(2m)}\cos(\beta t) + c_2 e^{-\gamma t/(2m)}\sin(\beta t). \longrightarrow 0 \text{ as } t \to \infty$$
This case, occuring when $\gamma < 2\sqrt{mk}$, is called **underdamping**.

Regardless of which situation we have, solutions die off exponentially as $t \to \infty$.

The implications of this on the forced (nonhomogeneous) DE

$$mu'' + \gamma u' + ku = F_0 \cos(\omega t),$$

are two-fold:

1. The general solution will contain a **transient** part $u_h(t)$ (coming from one of the three cases above) and a **steady-state** part $U(t)$.
2. What we pose for a particular solution (*a la* the method of undetermined coefficients) is just

$$u_p(t) = A\cos(\omega t) + B\sin(\omega t), \tag{4}$$

a type of *forced simple harmonic motion*, with no potential for a growing amplitude over time. One can solve for A, B to get

$$A = \frac{F_0(k - m\omega^2)}{(k - m\omega^2)^2 + \gamma^2\omega^2} \qquad \text{and} \qquad B = \frac{F_0\gamma\omega}{(k - m\omega^2)^2 + \gamma^2\omega^2}. \tag{5}$$

It is common to see a function in the form (4) rewritten as

$$u_p(t) = R\cos(\omega t - \delta),$$

with

$$R = \sqrt{A^2 + B^2}, \qquad \cos\delta = \frac{A}{R}, \qquad \text{and} \qquad \sin\delta = \frac{B}{R}. \tag{6}$$

This may be justified using the trigonometric identity

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta,$$

as this means

$$R\cos(\omega t - \delta) = R\cos\delta\cos(\omega t) + R\sin\delta\sin(\omega t).$$

Using the relations (6) on our expressions (5), we obtain particular solution

$$u_p(t) = \frac{F_0}{\Delta}\cos(\omega t - \delta),$$

with

$$\Delta = \sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}, \quad \cos\delta = \frac{m(\omega_0^2 - \omega^2)}{\Delta}, \quad \sin\delta = \frac{\gamma\omega}{\Delta}, \quad \text{and} \quad \omega_0 = \sqrt{\frac{k}{m}}.$$

Recalling that this particular solution represents the steady-state of the motions of our spring–mass assembly, we see that our assembly eventually settles into a delayed (and rescaled) version of the forcing term $F_0\cos(\omega t)$. While the amplitude F_0/Δ cannot grow with time, one can (in the underdamped case) *tune* the forcing frequency to the value $\omega = \omega_{\text{max}}$, where

$$\omega_{\text{max}}^2 := \omega_0^2 - \frac{\gamma^2}{2m^2} = \omega_0^2\left(1 - \frac{\gamma^2}{2mk}\right),$$

which maximizes this amplitude. In the case of small damping ($\gamma \ll 1$) we get vibrations with amplitude $F_0/\Delta \approx F_0/(\gamma\omega_0)$ that are far greater than the amplitude F_0 of the forcing function. Some people (the authors of our text among them) call this *resonance* as well, though it is a somewhat different concept than what was meant by the term above.

Electric circuits

Consider the electric circuit pictured at right. Here L, R, C are constants representing the **inductance**, **resistance** and **capacitance**, respectively. Let $Q(t)$ represent the total charge on the capacitor at time t, and $E(t)$ be the *impressed voltage*. One may reason (see the text, pp. 201–202 for some details) that the governing DE model is

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} \;=\; E(t).$$

The main point I wish to make is that the "features" we studied for a spring–mass assembly have analogues in the case of a simple RLC series circuit.