Math 231, Thu 8-Apr-2021 -- Thu 8-Apr-2021
Differential Equations and Linear Algebra
Spring 2021

Thursday, April 08th 2021

Due:: WW 4.1higherOrderDEs at 11 pm

Other calendar items

Thursday, April 8th 2021

Wk 10, Th
Topic:: Nonhomogeneous linear DEs

2^{nd} order linear

$$\underbrace{a_0(t)y'' + a_1(t)y' + a_2(t)y}_{\text{operation on } y \text{ - call it } L} = f(t)$$

reason for calling DE nonhomog.

$$\underline{Ly} = a_0(t)y'' + a_1(t)y' + a_2(t)y$$

L acting on y

Can write $\quad L = a_0(t)\frac{d^2}{dt^2} + a_1(t)\frac{d}{dt} + a_2(t)\,I$

Find fn. y such that $Ly = f$

Fact: L (as defined above) has the property that

$$\boxed{L(\underbrace{cy_1 + dy_2}_{\text{linear comb.}}) = c\,Ly_1 + d\,Ly_2}$$

(So L is said to be linear)

Linear Nonhomogeneous DEs

Form of L if
• nth order linear DE
• Coeff. of $y^{(n)}$ is 1.

We return now to the study of DEs of the form $L[y] = g$, where

$$L := \frac{d^n}{dt^n} + p_1(t)\frac{d^{n-1}}{dt^{n-1}} + + \cdots + p_{n-1}(t)\frac{d}{dt} + p_n(t), \tag{1}$$

with $g(t) \neq 0$. Earlier it was said that the paradigm we follow for solving such problems is

- Solve (i.e., find the general solution for) the homogeneous version of the problem. We will denote this *complementary solution* by $y_h(t)$ (or, if I slip up and call it $y_c(t)$ sometimes, know that I am referring to the same thing).
- Then use some means, perhaps simply a good guess, to find a *single* (particular) solution $y_p(t)$ of the full/original problem, and put the two answers together to get a general solution

$$y(t) = y_h(t) + y_p(t).$$

Finding this has been our focus so far in Ch. 4.
Solve: $Ly = 0$

It is a general solution because all solutions of $L[y] = g$ take this form.

While it is often difficult to find y_h, the general solution of $L[y] = 0$, we have a pretty good idea how to find it when the operator L has constant coefficients. The new issue is determining the single solution $y_p(t)$ of the original (nonhomogeneous) problem.

We will investigate two methods for finding a particular solution $y_p(t)$. The first could be called *making an educated guess*, but instead is called the **method of undetermined coefficients**. Its use is highly dependent on the form of the inhomogeneity $g(t)$. The other method is more analytical, requiring less in the way of good "intuition", but requires more in the way of technical calculations; it is called **variation of parameters**.

Now want to find y_p — two approaches
• Variation of params – formula involving integration
 – adaptable to very diverse settings
• Undetermined coeffs – no integrals to calculate
 – more limited in settings it applies to

Variation of parameters

- already have it for 1^{st}-order systems $\frac{d\vec{x}}{dt} = A\vec{x} + \vec{f}(t)$

$$\vec{x}_p(t) = \Phi(t) \int \Phi^{-1}(t)\, \vec{f}(t)\, dt$$

- Know: can convert a higher-order linear DE to a system

$$1\, y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = \underline{f(t)}$$

converted to a system looks like

$$\begin{bmatrix} y' \\ y'' \\ \vdots \\ y^{(n)} \end{bmatrix} = \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix} = \frac{d}{dt}\vec{x} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ & & & & \ddots & \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & -a_{n-3} & \cdots & -a_1 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \underline{f(t)} \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y' \\ \vdots \\ y^{(n-1)} \end{bmatrix}$$

- When we have solved, Ch.4 approach, $Ly = 0$, we have obtained n basis fns.
  - $e^{\lambda_1 t}, e^{\lambda_2 t}, \dots$ (n of these)
  - might appear $e^{\alpha t}\cos(\beta t)$, $e^{\alpha t}\sin(\beta t)$ (if complex roots)
  - might appear $te^{\lambda t}$, $t^2 e^{\lambda t}$ (if a root w/ algebraic mult > 1)

Soln. of homog. problem

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$$

where $y_1, y_2, \cdots, y_n$ are these basis fns,

Ex.) char. eqn.

$$(\lambda^2+1)(\lambda-3)^2 = 0$$

(original DE: same as $(\lambda^2-6\lambda+9)(\lambda^2+1)=0$

$$\lambda^4 - 6\lambda^3 + 10\lambda^2 - 6\lambda + 9 = 0$$

$$y^{(4)} - 6y''' + 10y'' - 6y' + 9\lambda = 0 .)$$

For solns. come from

$$\lambda = \pm i$$

$$\lambda = 3 \quad \text{(repeated)}$$

$$\Rightarrow \quad y_1(t) = \cos t$$

$$y_2(t) = \sin t$$

$$y_3(t) = e^{3t}$$

$$y_4(t) = te^{3t}$$

general soln. $y(t) = c_1 y_1 + c_2 y_2 + c_3 y_3 + c_4 y_4$

Aside:

In WebWork: If one basis soln. is $te^{2t}\cos(5t)$

Look at $e^{2t}\cos(5t)$ as having arisen from

a nonreal root: $\alpha = 2$, $\beta = 5$

Presence of t in $te^{2t}\cos(5t)$ is explained by

$$2 + 5i$$

being a double root.

Putting together my three bullet points: Observe that

If we know $y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$

and as well,

$$\begin{bmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{bmatrix} = \vec{x}_n(t) = \Phi(t) \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$$\Rightarrow \quad \Phi(t) = \begin{bmatrix} y_1(t) & y_2(t) & \cdots & y_n(t) \\ y_1'(t) & y_2'(t) & \cdots & y_n'(t) \\ \vdots & & & \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \cdots & y_n^{(n-1)}(t) \end{bmatrix}$$

and

y_p is the first coord. of $\vec{x}_p = \begin{bmatrix} y_p \\ y_p' \\ \vdots \\ y_p^{(n-1)} \end{bmatrix}$

and the 1^{st} coord of $\vec{x}_p$, by Cramer's Rule is taken from (just need the 1^{st} coord)

$$\Phi(t) \int \Phi^{-1} \vec{f}(t)\, dt$$

1^{st} coord of $\Phi^{-1}(t) \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ f(t) \end{bmatrix}}_{\vec{f}(t)}$ is $\dfrac{\begin{vmatrix} 0 & y_2 & y_3 & \cdots & y_n \\ 0 & y_2' & y_3' & \cdots & y_n' \\ \vdots & \vdots & & & \\ f(t) & y_2^{(n-1)} & y_3^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}}{\det(\Phi(t))}$

Need all such coords

Explains the presence of multiple

Upshot:

In case of 2^{nd}-order DEs

$$y_p(t) = y_1(t) \int \frac{\begin{vmatrix} 0 & y_2 \\ f(t) & y_2' \end{vmatrix}}{\det(\Phi(t))}\, dt$$

$$+ y_2(t) \int \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(t) \end{vmatrix}}{\det(\Phi(t))}\, dt$$

In the case of a 3^{rd} order linear DE $y''' + a_1 y'' + a_2 y' + a_3 y = f(t)$,

$$y_p(t) = y_1(t) \int \frac{\begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ f(t) & y_2'' & y_3'' \end{vmatrix}}{\det(\Phi(t))}\, dt + y_2(t) \int \frac{\begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & f(t) & y_3'' \end{vmatrix}}{\det(\Phi(t))}\, dt$$

$$+ y_3(t) \int \frac{\begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & f(t) \end{vmatrix}}{\det(\Phi(t))}\, dt,$$

where $\Phi(t) = \begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix}$

and y_1, y_2, y_3 are basis solns. of the homogeneous DE.

Undetermined coefficients

Your guesses should be tailored to the form of $g(t)$. Note that, by the linearity of the operator L, if $g(t) = g_1(t) + g_2(t) + \cdots + g_k(t)$, then the search for a particular solution $y_p(t)$ of

$$L[y](t) = g(t)$$

may be broken into the subproblems of finding a particular solution $Y_j(t)$ of

$$L[y](t) = g_j(t), \qquad \text{for } j = 1, \ldots, k.$$

That is, if we find Y_1 so that $L[Y_1] = g_1$, Y_2 so that $L[Y_2] = g_2$, etc., then $y_p(t) = Y_1(t) + Y_2(t) + \cdots + Y_k(t)$ satisfies $L[y_p] = g = g_1 + \cdots + g_k$.

It may well be that your intuition into differentiation (and DEs) is well enough attuned that you require little or no guidance on what kinds of guesses to make for a particular solution. This table, however, (mostly) lifted from p. 181 in the text, offers such guidance.

Form of $g_j(t)$	Form of particular soln $Y_j(t)$
$P_n(t) = a_0t^n + a_1t^{n-1} + \cdots + a_n$	$t^s(A_0t^n + A_1t^{n-1} + \cdots + A_n)$
$P_n(t)e^{\alpha t}$	$t^s(A_0t^n + A_1t^{n-1} + \cdots + A_n)e^{\alpha t}$
$P_n(t)e^{\alpha t}\sin(\beta t)$ or $P_n(t)e^{\alpha t}\cos(\beta t)$	$t^s[(A_0t^n + A_1t^{n-1} + \cdots + A_n)e^{\alpha t}\cos(\beta t)$
	$+(B_0t^n + B_1t^{n-1} + \cdots + B_n)e^{\alpha t}\sin(\beta t)]$
a form not in this list	no suggestions

The s that appears in the particular solution $Y_j(t)$ is the smallest nonnegative integer such that no term in $Y_j(t)$ is also found in the complementary solution $y_h(t)$.

Example 1:

Find particular solutions for

1. $y'' + 9y = 27t^2 - 18t + 51$
2. $y'' + 9y = (-9/2)e^{3t}$
3. $y'' + 9y = 27t^2 - 18t + 51 - 2e^{3t}$
4. $y'' - 10y' + 9y = 4e^t$
5. $y'' - 9y = e^{3t}$
6. $y'' - 9y = te^{3t}$
7. $y'' - 9y = e^{3t}\sin t$
8. $y'' - 2y' + 2y = e^t\sin t$

9. $y'' - 2y' + y = e^t$

■

If you are solving an IVP, you must *wait until you have the general solution to the full problem* $y_h(t) + y_p(t)$ before you apply the ICs.

Example 2: A nonhomogeneous linear IVP

Problem: Find the solution of the IVP

$$y'' - 2y' + y = e^t, \qquad y(0) = 1, \;\; y'(0) = -1.$$

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