$y'' + 9y = (-9/2)e^{3t}$

Nonhomog. 2^{nd}-order linear

$f(t) = -\frac{9}{2}e^{3t}$

1. Solve the homog. problem

$y'' + 9y = 0 \quad \longrightarrow \quad$ char. eqn. $\lambda^2 + 9 = 0$

$\longrightarrow$ char. values $\pm 3i$

$(\alpha = 0,\ \beta = 3)$

Pair of basis fns.

$$y_1(t) = \cos(3t), \quad y_2(t) = \sin(3t)$$

general soln.

$$y_h(t) = c_1 \cos(3t) + c_2 \sin(3t)$$

$$\Phi(t) = \begin{bmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{bmatrix}, \quad |\Phi(t)| = 3\cos^2(3t) - -3\sin^2(3t) = 3$$

$$y_p(t) = y_1(t) \int \frac{\begin{vmatrix} 0 & \sin(3t) \\ -\frac{9}{2}e^{3t} & 3\cos(3t) \end{vmatrix}}{\det(\Phi(t))} dt + y_2(t) \int \frac{\begin{vmatrix} \cos(3t) & 0 \\ -3\sin(3t) & -\frac{9}{2}e^{3t} \end{vmatrix}}{\det(\Phi(t))} dt$$

$$= \frac{1}{3} y_1(t) \int \frac{9}{2} e^{3t} \sin(3t)\, dt + y_2(t) \cdot \frac{1}{3} \int -\frac{9}{2} e^{3t} \cos(3t)\, dt$$

$$= \frac{3}{2} \cos(3t) \int e^{3t} \sin(3t)\, dt - \frac{3}{2} \sin(3t) \int e^{3t} \cos(3t)\, dt$$

Soln.

$$y_h(t) + y_p(t) = c_1 \cos(3t) + c_2 \sin(3t)$$

$$+ \frac{3}{2} \cos(3t) \int e^{3t} \sin(3t)\, dt - \frac{3}{2} \sin(3t) \int e^{3t} \cos(3t)\, dt$$

Undetermined coefficients

Your guesses should be tailored to the form of $g(t)$. Note that, by the linearity of the operator L, if $g(t) = g_1(t) + g_2(t) + \cdots + g_k(t)$, then the search for a particular solution $y_p(t)$ of

$$L[y](t) = g(t)$$

may be broken into the subproblems of finding a particular solution $Y_j(t)$ of

$$L[y](t) = g_j(t), \qquad \text{for } j = 1, \ldots, k.$$

That is, if we find Y_1 so that $L[Y_1] = g_1$, Y_2 so that $L[Y_2] = g_2$, etc., then $y_p(t) = Y_1(t) + Y_2(t) + \cdots + Y_k(t)$ satisfies $L[y_p] = g = g_1 + \cdots + g_k$.

It may well be that your intuition into differentiation (and DEs) is well enough attuned that you require little or no guidance on what kinds of guesses to make for a particular solution. This table, however, (mostly) lifted from p. 181 in the text, offers such guidance.

Nonhomog. $f(t)$	Form of $g_j(t)$	Form of particular soln $Y_j(t)$
polynomial $\rightarrow$	$P_n(t) = a_0t^n + a_1t^{n-1} + \cdots + a_n$	$t^s(A_0t^n + A_1t^{n-1} + \cdots + A_n)$
(poly.)(exp) $\rightarrow$	$P_n(t)e^{\alpha t}$	$t^s(A_0t^n + A_1t^{n-1} + \cdots + A_n)e^{\alpha t}$
(poly)(exp) $\cdot$ (sine cosine) $\rightarrow$	$P_n(t)e^{\alpha t}\sin(\beta t)$ or $P_n(t)e^{\alpha t}\cos(\beta t)$	$t^s[(A_0t^n + A_1t^{n-1} + \cdots + A_n)e^{\alpha t}\cos(\beta t)$ $+(B_0t^n + B_1t^{n-1} + \cdots + B_n)e^{\alpha t}\sin(\beta t)]$
	a form not in this list	no suggestions

The s that appears in the particular solution $Y_j(t)$ is the smallest nonnegative integer such that no term in $Y_j(t)$ is also found in the complementary solution $y_h(t)$.

Example 1:

Find particular solutions for

1. $y'' + 9y = 27t^2 - 18t + 51$
2. $y'' + 9y = (-9/2)e^{3t}$

$\}$ $y'' + 9y = 27t^2 - 18t + 51 - \frac{9}{2}e^{3t}$

3. $y'' + 9y = 27t^2 - 18t + 51 - 2e^{3t}$
4. $y'' - 10y' + 9y = 4e^t$
5. $y'' - 9y = e^{3t}$
6. $y'' - 9y = te^{3t}$
7. $y'' - 9y = e^{3t}\sin t$
8. $y'' - 2y' + 2y = e^t\sin t$

9. $y'' - 2y' + y = e^t$

$\blacksquare$

If you are solving an IVP, you must *wait until you have the general solution to the full problem* $y_h(t) + y_p(t)$ before you apply the ICs.

Example 2: A nonhomogeneous linear IVP

Problem: Find the solution of the IVP

$$y'' - 2y' + y = e^t, \qquad y(0) = 1, \;\; y'(0) = -1.$$

$\blacksquare$

see: (0^{th}-deg. poly)(exp.)

$$y'' - 2y' + 7y = 3e^{2t}$$

$$\underbrace{y'' - 2y' + 7y}_{Ly}$$

Propose $y_p = Ae^{2t}$ (another (0^{th} deg. poly) $\cdot$ (exp.))

$$y_p' = 2Ae^{2t}$$

$$y_p'' = 4Ae^{2t}$$

Insert into the nonhomog. DE

goal / target

$$4Ae^{2t} - 2(2Ae^{2t}) + 7(Ae^{2t}) = 3e^{2t}$$

$$7Ae^{2t} = 3e^{2t} \longrightarrow A = \frac{3}{7} \text{ works}$$

$$y_p(t) = \frac{3}{7}e^{2t}$$

Ex.

$$y'' + 9y = \underbrace{27t^2 - 18t + 51}_{2^{nd}\text{-deg poly.}}$$

To solve, 1^{st} solve homog problem $y'' + 9y = 0$

From earlier, got

$$y_h(t) = c_1 \cos(3t) + c_2 \sin(3t)$$

Propose: $y_p(t) = At^2 + Bt + C$ } insert into nonhomog.

$y_p' = 2At + B$

$y_p'' = 2A$

DE

	LHS		RHS (target)
	$2A + 9(At^2 + Bt + C)$	$=$	$27t^2 - 18t + 51$

Can equate coeffs for like terms

	LHS		RHS	
const.	$2A + 9C$	$=$	51	$\Rightarrow 6 + 9C = 51 \Rightarrow C = 5$
t^1	$9B$	$=$	-18	$\Rightarrow B = -2$
t^2	$9A$	$=$	27	$\Rightarrow A = 3$

$y_p(t) = 3t^2 - 2t + 5$

general soln: $y_h + y_p = c_1 \cos(3t) + c_2 \sin(3t) + 3t^2 - 2t + 5$.

Ex.)

$y'' - 9y = e^{3t} \sin t$

$y'' - 9y = 0 \longrightarrow \lambda^2 - 9 = 0$

$(\lambda + 3)(\lambda - 3)$

$y_h(t) = c_1 e^{-3t} + c_2 e^{3t}$

$f(t) = e^{3t} \sin t \qquad (\exp)(\sin)$

natural $\quad y_p(t) = (A \cos t + B \sin t) e^{3t}$