Today

$$y'' + 2by' + \omega_0^2 y = f_0 \cos(\omega t) \qquad \underline{(\star)}$$

Note $$mu'' + \gamma u' + ku = F_0 \cos(\omega t)$$

can be transformed to $(\star)$ by

- dividing by m
- defining $2b = \frac{\gamma}{m}$ $\left(\text{set } b = \frac{\gamma}{2m}\right)$
- defining $f_0 = F_0/m$.

Note: $2b$, ω_0^2 both positive

So by work from last week, y_h is transient $(\to 0 \text{ as } t \to \infty)$

Thus, y_p is the "steady state"

$$y(t) = y_h(t) + y_p(t) \longrightarrow y_p(t)$$

Task 1: Find y_p, particular soln of

$$y'' + 2by' + \omega_0^2 y = \underbrace{f_0 \cos(\omega t)}$$

fits requirements of undetermined coeffs
- RHS has acceptable form
- constant coeffs

$$y_p(t) = A\cos(\omega t) + B\sin(\omega t)$$

$$y_p' = -\omega A \sin(\omega t) + \omega B \cos(\omega t)$$

$$y_p'' = -\omega^2 A \cos(\omega t) - \omega^2 B \sin(\omega t)$$

$$\text{LHS of DE} = \cos(\omega t)\left[-\omega^2 A + 2\omega b B + \omega_0^2 A\right] + \sin(\omega t)\left[-\omega^2 B - 2\omega b A + \omega_0^2 B\right]$$

$$\overset{\text{target}}{=} f_0 \cos(\omega t)$$

Equate coeffs:

terms	LHS		target
$\cos(\omega t)$	$-\omega^2 A + 2\omega b B + \omega_0^2 A$	$=$	f_0
$\sin(\omega t)$	$-\omega^2 B - 2\omega b A + \omega_0^2 B$	$=$	0

} two eqn. in 2 unknowns A, B

In matrix form

$$\begin{bmatrix} \omega_0^2 - \omega^2 & 2\omega b \\ -2\omega b & \omega_0^2 - \omega^2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} f_0 \\ 0 \end{bmatrix}$$

Steps to get here

$$\begin{bmatrix} f_0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\omega^2 A + 2\omega b B + \omega_0^2 A \\ -\omega^2 B - 2\omega b A + \omega_0^2 B \end{bmatrix} = \begin{bmatrix} \omega_0^2 A - \omega^2 A \\ -2\omega b A \end{bmatrix} + \begin{bmatrix} 2\omega b B \\ \omega_0^2 B - \omega^2 B \end{bmatrix}$$

$$= A \begin{bmatrix} \omega_0^2 - \omega^2 \\ -2\omega b \end{bmatrix} + B \begin{bmatrix} 2\omega b \\ \omega_0^2 - \omega^2 \end{bmatrix}$$

Note

$$\begin{vmatrix} \omega_0^2 - \omega^2 & 2\omega b \\ -2\omega b & \omega_0^2 - \omega^2 \end{vmatrix} = \left(\omega_0^2 - \omega^2\right)^2 + 4\omega^2 b^2 =: \Delta$$

By Cramer's Rule

$$A = \frac{\begin{vmatrix} f_0 & 2\omega b \\ 0 & \omega_0^2 - \omega^2 \end{vmatrix}}{\Delta} = \frac{(\omega_0^2 - \omega^2) f_0}{\Delta}$$

$$B = \frac{\begin{vmatrix} \omega_0^2 - \omega^2 & f_0 \\ -2\omega b & 0 \end{vmatrix}}{\Delta} = \frac{2\omega b f_0}{\Delta}$$

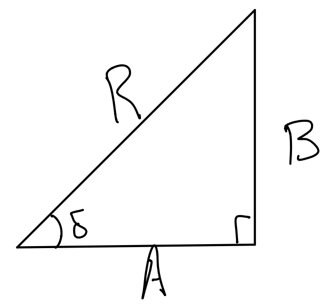
Have found our steady state soln.

$$y_p = A\cos(\omega t) + B\sin(\omega t)$$

w/ these for A, B.

Rewrite steady state y_p in form

$$y_p(t) = R\cos(\omega t - \delta)$$



$$R = \sqrt{A^2 + B^2} = \frac{\sqrt{(\omega_0^2 - \omega^2)^2 f_0^2 + 4\omega^2 b^2 f_0^2}}{\sqrt{\Delta^2}} = \sqrt{\frac{f_0^2}{\Delta^2}} \cdot \sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2 b^2}$$

$$= \sqrt{\frac{f_0^2}{\Delta^2} \cdot \Delta} = \frac{f_0}{\sqrt{\Delta}}$$

Diagram tells us

$$\sin\delta = \frac{B}{R} = \frac{2\omega b f_0 / \Delta}{f_0 / \sqrt{\Delta}} = \frac{2\omega b}{\sqrt{\Delta}}$$

$$\cos\delta = \frac{A}{R} = \frac{(\omega_0^2 - \omega^2) f_0 / \Delta}{f_0 / \sqrt{\Delta}} = \frac{\omega_0^2 - \omega^2}{\sqrt{\Delta}} \Rightarrow \delta = \cos^{-1}\left(\frac{\omega_0^2 - \omega^2}{\sqrt{\Delta}}\right)$$

$$\tan\delta = \frac{B}{A} = \frac{2\omega b f_0 / \Delta}{(\omega_0^2 - \omega^2) f_0 / \Delta} = \frac{2\omega b}{\omega_0^2 - \omega^2} \Rightarrow \delta = \tan^{-1}\left(\frac{2\omega b}{\omega_0^2 - \omega^2}\right)$$