Laplace Transform

Vibrations $\quad mu'' + \gamma u' + ku = f(t)$

$f(t)$ has variously been

0

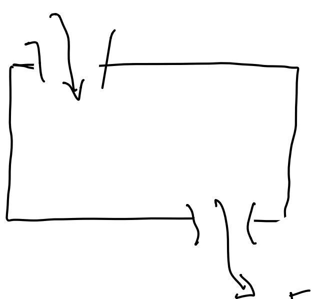
$F_0 \cos(\omega t)$ (simple, periodic forcing fn)

exponential, polynomial, sine/cosine (or combinations of these)

But what of other sorts of forcing fns?

General opinion: Laplace transform are more efficient then ones learned already when forcing fns. are more complicated.

$f(t)$: accept t as input



$F(s) = \mathcal{L}\{f(t)\}$

new variable: s

$$\boxed{\mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st} f(t)\, dt}$$

(definition of Laplace transf. of f)

Required of f:

- has to be defined on $(0, \infty)$ (i.e. $(0, \infty)$ should be in domain of f)
- isn't allowed to grow too fast so that this integral diverges
  (f must be of "exponential order")

Build a catalog of

time side $f(t)$	frequency side $\mathcal{L}\{f(t)\}(s)$
0	0
1	$1/s, \quad s>0$
t	$1/s^2, \quad s>0$
t^2	$2!/s^3, \quad s>0$
t^n, $n>0$, integer	$n!/s^{n+1}, \quad s>0$
e^{at}	?

$f(t) = 3t^2 - 2t + 5$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}\left(3t^2 - 2t + 5\right)dt$$

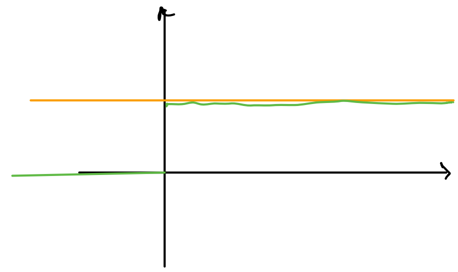
$$= 3\,\mathcal{L}\{t^2\} - 2\,\mathcal{L}\{t\} + 5\,\mathcal{L}\{1\}$$

$$= 3 \cdot \frac{2}{s^3} - 2 \cdot \frac{1}{s^2} + 5 \cdot \frac{1}{s}.$$

$f(t) = 1$

$$\mathcal{L}\{f\} = \int_0^\infty e^{-st} \cdot 1 \, dt$$

result is same for



$$= \lim_{A \to \infty} \int_0^A e^{-st} \, dt$$

$$= \lim_{A \to \infty} \left[\frac{-1}{s} e^{-st} \right]_0^A$$

$$= \lim_{A \to \infty} \left(-\frac{1}{s} e^{-sA} - -\frac{1}{s} \right) = \frac{1}{s} \cdot \lim_{A \to \infty} \underbrace{\left(1 - e^{-sA}\right)}$$

when $s > 0$, goes to 1

when $s < 0$, diverges

So, if $s > 0$, get

$$= \frac{1}{s}.$$

Ex.

$$\mathcal{L}\{t\}(s) = \int_0^\infty e^{-st} \cdot t \, dt = \lim_{A \to \infty} \int_0^A t e^{-st} \, dt$$

Aside

$$\int_0^A t e^{-st} \, dt = \int_0^A u \, dv = uv\Big|_0^A - \int_0^A v \, du$$

Call $u = t$, $du = dt$

$dv = e^{-st} dt$, $v = -\frac{1}{s} e^{-st}$

$$= -\frac{1}{s} t e^{-st}\Big|_0^A - \int_0^A -\frac{1}{s} e^{-st} \, dt$$

$$= -\frac{1}{s} A e^{-sA} - 0 + \frac{1}{s} \int_0^A e^{-st} \, dt$$

$$= -\frac{1}{s} A e^{-sA} - \frac{1}{s^2} \left[e^{-st} \right]_0^A$$

$$= -\frac{1}{s} A e^{-sA} - \frac{1}{s^2}\left(e^{-sA} - 1\right)$$

$$\mathcal{L}\{t\}(s) = \lim_{A\to\infty}\left[\underbrace{-\frac{1}{s} A e^{-sA}}_{\substack{\to 0 \text{ as} \\ A\to\infty}} - \frac{1}{s^2}\left(\underbrace{e^{-sA}}_{\substack{\to 0 \text{ as} \\ A\to\infty \\ \text{if } s>0}} - 1\right)\right]$$

Note: For convergence we again need $s > 0$

$$= \frac{1}{s^2}$$