Math 231, Mon 26-Apr-2021 -- Mon 26-Apr-2021
Differential Equations and Linear Algebra
Spring 2021

B. $\mathcal{L}\{H(t-a)f(t-a)\} = e^{-as}F(s)$

A. $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$

right a units

Monday, April 26th 2021

Wk 13, Mo
Topic:: More with shift theorems
Topic:: Solving IVPs

Shift theorems:
- Finding LT of H(t-c)f(t), no obvious shift in the function f
  example: L{ H(t-2)(t^2+3t-1) }

- Finding ILT of exp(-as)F(s)
  example: Find ILT of e^{-2s} (3s+4)/(s^2 + 6s + 13)

Effect of LT on derivatives of a function

Use of LT to solve IVPs
 general: ay'' + by' + cy = g(t), y(0) = k0, y'(0) = k1
  Note one can split into two problems
   (1) ay'' + by' + cy = 0, y(0) = k0, y'(0) = k1
   (2) ay'' + by' + cy = g(t), y(0) = 0, y'(0) = 0
  Make case for using LT only on problem (2); use Chapter 4 methods on (1)

Didn't get to this

 example:

Ex.] $\mathcal{L}\{H(t-2)(t^2+3t-1)\}$

Not mandatory to use shift thm B (not convenient either).

Can do directly from defn.

$$\mathcal{L}\{H(t-2)(t^2+3t-1)\} = \int_0^\infty e^{-st} H(t-2)(t^2+3t-1)\,dt$$

$$= \int_2^\infty e^{-st}(t^2+3t-1)\,dt$$

(improper, requires integration by parts $\times 2$)

How would this go using Thm. B?

First have to determine the function, $f(t)$, which when shifted 2 units to the right, becomes t^2+3t-1.

Some options

1. Coefficient-matching
2. Taylor expansion
3. Try left-shifting the correct amount

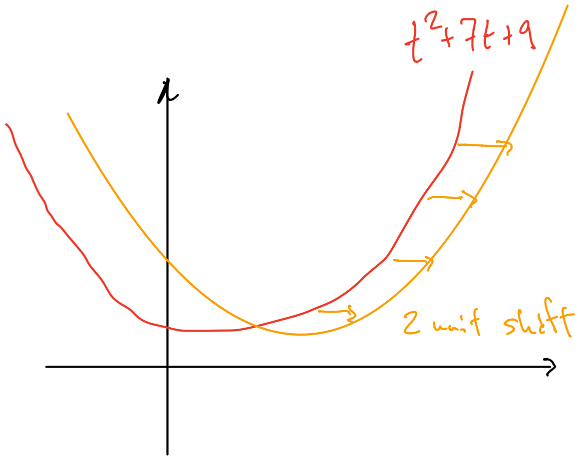
To left-shift t^2+3t-1 2 units,

do $t \mapsto t+2$

$$(t+2)^2 + 3(t+2) - 1$$

$$= t^2+4t+4+3t+6-1$$

$$= t^2+7t+9$$



g is red fn

Now can call

$$H(t-2)(t^2+3t-1) = H(t-2)\,g(t-2)$$

and Thm. B becomes applicable

$$\mathcal{L}\{H(t-2)(t^2+3t-1)\} = \mathcal{L}\{H(t-2)\,g(t-2)\}$$

$$\overset{\text{Thm. B}}{=} e^{-2s}\cdot\mathcal{L}\{t^2+7t+9\}$$

$$= e^{-2s}\left(\frac{2!}{s^3} + 7\frac{1}{s^2} + 9\cdot\frac{1}{s}\right)$$

Coming from freq. to time side involving shift thm.

$$F(s) = e^{-5s}\cdot\frac{2s+4}{s^2+6s+13}$$

Want $f(t)$ so that $\mathcal{L}\{f(t)\}(s) = F(s)$.

First: Seeing an exponential in $F(s)$ — think Thm. B.

Already thinking

$$f(t) = H(t-5)\ \underline{\qquad\qquad}$$

something called $g(t-5)$

Now focus on what fn. $g(t)$ has a L.T. $\dfrac{2s+4}{s^2+6s+13}$

Since $s^2+9s+13$ is irreducible,

what must go here to maintain my numerator

$$\frac{2s+4}{s^2+6s+13} = \frac{2s+4}{(s+\underbrace{3)^2}_{s^2+6s+9}+4} = \frac{2(s+3)+ -2}{(s+3)^2+4}$$

$s \mapsto s+3$ is a shift on freq. side suggesting Thm A is applicable

$$= \left.\frac{2s-2}{s^2+4}\right|_{s \mapsto s-(-3)}$$

Because

$$\frac{2s-2}{s^2+4} = 2 \cdot \frac{s}{s^2+4} - \frac{2}{s^2+4}$$

$$= 2\,\mathcal{L}\{\cos(2t)\} - \mathcal{L}\{\sin(2t)\}$$

the shifted case

by Thm. A

$$\mathcal{L}^{-1}\left\{\left.\frac{2s-2}{s^2+4}\right|_{s \mapsto s-(-3)}\right\} = \underbrace{e^{-3t}\cdot\left[2\cos(2t)-\sin(2t)\right]}_{\text{our } g(t)}$$

Earlier we had determined our answer would look like

$$H(t-5)\, g(t-5)$$

$$= H(t-5)\, \underbrace{e^{-3(t-5)}\left[2\cos(2(t-5)) - \sin(2(t-5))\right]}_{g(t-5)}$$

Connection of L.T. to DEs: How L.T. works on derivatives.

Say you have agreed to call $\mathcal{L}\{f(t)\} = F(s) = \boxed{\int_0^\infty e^{-st} f(t)\, dt}$

Now you need $\mathcal{L}\{f'(t)\}$.

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\, dt$$

Use int. by parts

$u = e^{-st}$ | $du = -se^{-st}\, dt$

$dv = f'(t)\, dt$ | $v = f(t)$

$$= u(t)\, v(t)\Big|_0^\infty - \int_0^\infty v(t)\, u'(t)\, dt$$

$$= e^{-st} f(t)\Big|_0^\infty - \int_0^\infty -se^{-st} f(t)\, dt$$

$$= \underbrace{\left(\lim_{t\to\infty} e^{-st} f(t)\right)}_{\text{for nice enough fns., } s>0, \text{ will be } 0} - e^{0}\cdot f(0) + s\boxed{\int_0^\infty e^{-st} f(t)\, dt}$$

$$\mathcal{L}\{f'\} = sF(s) - f(0).$$

Can iterate this

$$\mathcal{L}\{f''(t)\} = s\,\mathcal{L}\{f'(t)\} - f'(0)$$

$$= s\left[sF(s) - f(0)\right] - f'(0)$$

$$= s^2 F(s) - s f(0) - f'(0)$$

$$\mathcal{L}\{f'''\} = \cdots = s^3 F(s) - s^2 f(0) - s f'(0) - f''(0)$$

Applicable to IVP

$$ay'' + by' + cy = f(t), \qquad \text{subj. to ICs} \quad y(0) = y_0, \ y'(0) = y_1$$

Take L.T. of both sides

$$\mathcal{L}\{ay'' + by' + cy\} = \mathcal{L}\{f(t)\}$$

LHS

$$\underline{a\,\mathcal{L}\{y''\}} + \underline{b\,\mathcal{L}\{y'\}} + c\,\mathcal{L}\{y\}$$

call this Y

$$a\left[s^2 Y - s y(0) - y'(0)\right] + b\left[sY - y(0)\right] + cY = F(s)$$

One can use algebra to solve for Y.

Then $y(t) = \mathcal{L}^{-1}\{Y(s)\}$