Math 231, Wed 28-Apr-2021 -- Wed 28-Apr-2021
Differential Equations and Linear Algebra
Spring 2021

Wednesday, April 28th 2021

Wk 13, We
Topic:: Dirac delta function
HW:: HC07

Use of LT to solve IVPs
 general: ay'' + by' + cy = g(t), y(0) = k0, y'(0) = k1
   Note one can split into two problems
     (1) ay'' + by' + cy = 0, y(0) = k0, y'(0) = k1
     (2) ay'' + by' + cy = g(t), y(0) = 0, y'(0) = 0
   Make case for using LT only on problem (2); use Chapter 4 methods on (1)

Example: From Exercise 5.3.2
  Solve y'' + 7y' + 12y = 3exp(-2t), y(0)=1, y'(0)=3

$$ay'' + by' + cy = f(t), \quad y(0) = y_0, \ y'(0) = y_1$$

From last time, after taking L.T. of both sides

$$a\left[s^2 Y - s y(0) - y'(0)\right] + b\left[sY - y(0)\right] + cY = F(s)$$

$$a\left[s^2 Y - s y_0 - y_1\right] + b\left[sY - y_0\right] + cY = F(s)$$

Now solve for $Y = Y(s)$

$$as^2Y - asy_0 - ay_1 + bsY - by_0 + cY = F$$

Keep Y terms on one side

$$Y(as^2 + bs + c) = asy_0 + ay_1 + by_0 + F$$

Now divide

$$Y = Y(s) = \frac{asy_0 + ay_1 + by_0}{as^2 + bs + c} + \frac{F(s)}{as^2 + bs + c}$$

Now get $y(t)$ using ILT

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{asy_0 + ay_1 + by_0}{as^2 + bs + c} + \frac{F(s)}{as^2 + bs + c}\right\}$$

$$= \underbrace{\mathcal{L}^{-1}\left\{\frac{asy_0 + ay_1 + by_0}{as^2 + bs + c}\right\}}_{\text{solves (1)}} + \underbrace{\mathcal{L}^{-1}\left\{\frac{F(s)}{as^2 + bs + c}\right\}}_{\text{solves (2)}}$$

Solves

(1) $ay'' + by' + cy = 0, \quad y(0) = y_0, \; y'(0) = y_1$

(2) $ay'' + by' + cy = f(t), \quad y(0) = 0, \; y'(0) = 0$

Solves

Ex.] $y'' + 7y' + 12y = 3e^{-2t}$, $y(0) = 1$, $y'(0) = 3$

Based on the above, divide into 2 probs.

(1) homogeneous, keeps the ICs

$$y'' + 7y' + 12y = 0, \quad y(0) = 1, \; y'(0) = 3$$

(2) nonhomog., w/ zeroed ICs

$$y'' + 7y' + 12y = 3e^{-2t}, \quad y(0) = 0, \; y'(0) = 0$$

(1) has characteristic eqn.

$$\lambda^2 + 7\lambda + 12 = 0$$

$$(\lambda + 4)(\lambda + 3) = 0 \quad \rightarrow \text{ char. vals. } \lambda = -3, -4$$

$\rightarrow$ individual solns. (fndl set)

$$e^{-3t}, \; e^{-4t}$$

general soln

$$y_h(t) = c_1 e^{-3t} + c_2 e^{-4t}$$

$$y_h'(t) = -3c_1 e^{-3t} - 4c_2 e^{-4t}$$

Applying ICs

$$c_1 e^0 + c_2 e^0 = y(0) = 1$$

$$-3c_1e^0 - 4c_2e^0 = y'(0) = 3$$

$$\begin{bmatrix} 1 & 1 \\ -3 & -4 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$c_1 = \frac{\begin{vmatrix} 1 & 1 \\ 3 & -4 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -3 & -4 \end{vmatrix}} = \frac{-7}{-1} = 7$$

$$c_2 = \frac{\begin{vmatrix} 1 & 1 \\ -3 & 3 \end{vmatrix}}{-1} = \frac{6}{-1} = -6$$

(1) has soln.

$$7e^{-3t} - 6e^{-4t}.$$

(2), if we use L.T. methods is

$$Y(s) = \frac{\mathcal{L}\{3e^{-2t}\}}{s^2 + 7s + 12} = \frac{3 \cdot \frac{1}{s+2}}{s^2 + 7s + 12}$$

$$= \frac{3}{(s+3)(s+4)(s+2)}$$

$$= \frac{A}{s+3} + \frac{B}{s+4} + \frac{C}{s+2}$$

Find A, B, C, equate

$$\frac{3}{(s+3)(s+4)(s+2)} = \frac{A}{s+3} + \frac{B}{s+4} + \frac{C}{s+2}$$

Mult. by Common Denominator: $(s+3)(s+4)(s+2)$

$$3 = A(s+4)(s+2) + B(s+3)(s+2) + C(s+4)(s+3)$$

Note that, since the sides must be equal for all values of s (must be an identity), choose favorable values:

@ $s=-3$: eqn collapses

$$3 = A(-3+4)(-3+2) + 0 + 0$$
$$= -A \Rightarrow A = -3$$

@ $s=-4$:

$$3 = 0 + B(-4+3)(-4+2) + 0$$
$$\Rightarrow B = 3/2$$

@ $s=-2$:

$$3 = 0 + 0 + C(-2+4)(-2+3)$$
$$\Rightarrow C = 3/2$$

Gain? Soln. to (2)

$$Y(s) = \frac{3}{(s+4)(s+3)(s+2)} = \frac{3/2}{s+4} - \frac{3}{s+3} + \frac{3/2}{s+2}$$

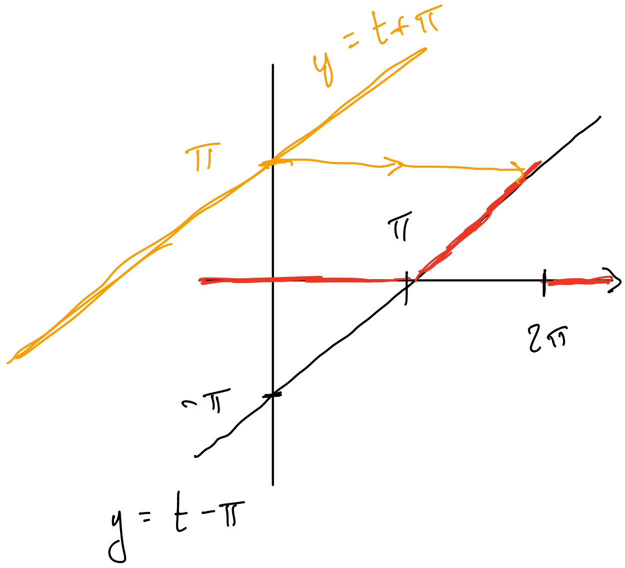
$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{3}{2}e^{-4t} - 3e^{-3t} + \frac{3}{2}e^{-2t}$$

Original problem's solution: Sum of solns. to (1) and (2)

$$y(t) = \underbrace{7e^{-3t} - 6e^{-4t}}_{\text{from (1)}} + \underbrace{\frac{3}{2}e^{-4t} - 3e^{-3t} + \frac{3}{2}e^{-2t}}_{\text{from (2)}}$$

1 (d)

$$f(t) = \begin{cases} 0, & t < \pi \\ t - \pi, & \pi \le t < 2\pi \\ 0, & t \ge 2\pi \end{cases}$$



$$= H(t-\pi)(t-\pi) - H(t-2\pi)(t-\pi)$$

base fn.

$y = t$

(now shifted to $t - \pi$)

$t + \pi$ base fn.

when shifted right 2π units becomes

$t - \pi$

base fn.

t

has L.T.

$$\frac{1}{s^2}$$

base fn. $t+\pi$

has L.T.

$$\frac{1}{s^2} + \frac{\pi}{s}$$