Ex.)

$$y'' + 10y' + 25y = f(t)$$

(a) Find the transfer fn.

Recall: In the process of using L.T. on this sort, got

$$\left[s^2 + 10s + 25 \right] Y(s) = F(s) + \text{other terms}?$$

present if ICs are not all zeroed.

$$\Rightarrow \quad Y(s) = \underbrace{\frac{1}{s^2 + 10s + 25}}_{\text{transfer fn.}} \cdot F(s)$$

(Case when ICs are zeroed)

(b) If ICs are zeroed: $y(0) = 0$, $y'(0) = 0$,

then write the solution of this IVP as a convolution integral.

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{1}{s^2 + 10s + 25} \cdot F(s) \right\}$$

$$= \mathcal{L}^{-1}\left\{ \frac{1}{s^2 + 10s + 25} \right\} * \underbrace{\mathcal{L}^{-1}\{F(s)\}}_{= f(t)}$$

operation on time side

$$\frac{1}{s^2 + 10s + 25} = \frac{1}{(s+5)^2} = \left. \frac{1}{s^2} \right|_{s \mapsto s - (-5)}$$

$$\mathcal{L}^{-1}\left\{ \frac{1}{s^2 + 10s + 25} \right\} = e^{-5t} \cdot t \qquad \text{(by Shift Theorem A)}$$

So $$y(t) = \underbrace{\left(te^{-5t} \right)}_{\text{impulse response}} * f(t)$$

$$= \int_0^t \underbrace{f(t-w)}_{\text{fn. of } t-w} \cdot \underbrace{w e^{-5w}}_{\text{fn. of } w} \, dw$$

Ex.

$$y'' + 4y' + 5y = \underbrace{3e^{-2t}\cos t}_{\text{lends itself to undet. coeffs}}, \quad y(0) = 1, \quad y'(0) = -1$$

Ch. 4 approach

1. Solve $y'' + 4y' + 5y = 0$

$\longrightarrow$ char. eqn $\lambda^2 + 4\lambda + 5 = 0$

$$\lambda = -2 \pm i \implies \begin{aligned} \alpha &= -2 \\ \beta &= 1 \end{aligned}$$

Both
$$\left.\begin{aligned} y_1 &= e^{-2t}\cos t \\ y_2 &= e^{-2t}\sin t \end{aligned}\right\} \text{ solve homog. problem}$$

$$y_h(t) = c_1 e^{-2t}\cos t + c_2 e^{-2t}\sin t \quad \checkmark$$

2. Need y_p – use Undet. Coeffs.

Natural to propose
$$y_p = \left[A e^{-2t}\cos t + B e^{-2t}\sin t\right] t$$

introduced to avoid similar terms in y_h

Plugging this y_p (along w/ y_p', y_p'') into LHS of the DE

$$y_p'' + 4y_p' + 5y_p = \text{combine terms (much algebra)}$$

$$= 2B\cos(t)\cdot e^{-2t} - 2A\sin(t)\cdot e^{-2t}$$

$$\stackrel{\text{target}}{=} 3e^{-2t}\cos t$$

$$\rightarrow \quad A = 0, \qquad B = \frac{3}{2}$$

So full general soln.

$$y(t) = y_h + y_p = c_1 e^{-2t}\cos t + c_2 e^{-2t}\sin t + \frac{3}{2}te^{-2t}\sin t$$

$$\Rightarrow \quad y'(t) = -2c_1e^{-2t}\cos t - c_1e^{-2t}\sin t$$
$$- 2c_2e^{-2t}\sin t + c_2e^{-2t}\cos t$$
$$+ \frac{3}{2}e^{-2t}\sin t - 3te^{-2t}\sin t + \frac{3}{2}te^{-2t}\cos t$$

ICs

$$1 = y(0) = c_1$$
$$-1 = y'(0) = -2c_1 + c_2 = -2 + c_2 \Rightarrow c_2 = 1$$

Ch. 4 approach finished, have soln.

$$y(t) = e^{-2t}\cos t + e^{-2t}\sin t + \frac{3}{2}te^{-2t}\sin t$$

For L.T. (Chapter 5 methods)

$$y'' + 4y' + 5y = 3e^{-2t}\cos t, \qquad y(0) = 1, \quad y'(0) = -1$$

Can split into 2 probs.

(1) $y'' + 4y' + 5y = 0, \qquad y(0) = 1, \quad y'(0) = 1$

Already know $y_h(t) = c_1 e^{-2t}\cos t + c_2 e^{-2t}\sin t$

Valid to find c_1, c_2 directly from ICs (without knowing soln. to (2)) because the particular soln. we get from (2) will solve a problem w/ zeroed ICs.

(2) $y'' + 4y' + 5y = 3e^{-2t}\cos t, \qquad y(0) = 0, \quad y'(0) = 0$

Do like (?) one of our earliest probs. today