From Kapitula's book

$$y'' - y = \frac{6}{1+e^t}$$

Note
- linear, 2^{nd}-order
- nonhomogeneous

NH term $f(t) = \dfrac{6}{1+e^t}$

Not exponential (Ae^{kt})

Not sine or cosine

Not polynomial: linear combs. of nonneg. integer powers of t

$1, t, t^2, t^3, \ldots$

y_p not best found using Undet Coeffs (so Variation of Params)

y_h = gen'l soln. to $y'' - y = 0$

$$\lambda^2 - 1 = 0 \Rightarrow (\lambda+1)(\lambda-1) = 0$$

$$\Rightarrow \lambda = \pm 1$$

Fund'l set of solns. $y_1 = e^{-t}$, $y_2 = e^t$

$$\text{Wronskian} = |\Phi(t)| = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-t} & e^t \\ -e^{-t} & e^t \end{vmatrix} = 1 - (-1) = 2$$

Variation of Params formula

$$y_p = y_1 \cdot \underbrace{\int \frac{\begin{vmatrix} 0 & y_2 \\ f & y_2' \end{vmatrix}}{|\Phi(t)|}}\,dt + y_2 \underbrace{\int \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f \end{vmatrix}}{|\Phi(t)|}}\,dt$$

Two components of product $\Phi(t)\begin{bmatrix} 0 \\ f \end{bmatrix}$ via Cramer's Rule

$$= \underbrace{e^{-t}}_{y_1} \int \frac{1}{2} \begin{vmatrix} 0 & e^{t} \\ \frac{6}{1+e^{t}} & e^{t} \end{vmatrix} dt + \underbrace{e^{t}}_{y_2} \int \frac{1}{2} \begin{vmatrix} e^{-t} & 0 \\ -e^{-t} & \frac{6}{1+e^{t}} \end{vmatrix}$$

$$= \frac{1}{2} e^{-t} \underbrace{\int \frac{6e^{t}}{1+e^{t}}\, dt} + \frac{1}{2} e^{t} \int \frac{6e^{-t}}{1+e^{t}}\, dt$$

substitute for this

$u = 1 + e^{t}$

$du = e^{t}\, dt$

$$\int \frac{6e^{t}}{1+e^{t}}\, dt = \int \frac{6\, du}{u} = 6 \ln|u| = 6 \ln|1+e^{t}|$$

$+ C$?

not needed, since arb. constants already in y_h

$$\int \frac{6e^{-t}}{1+e^{t}}\, dt$$

$$\frac{6e^{-t}}{1+e^{t}} \cdot \frac{e^{t}}{e^{t}} = \frac{6}{e^{t}+e^{2t}}$$ better?

From #23 in Ch. 5 WWork

$$Y(s) = \frac{1/s^2 - e^{-2s}\left(1/s^2 + 2/s\right)}{s^2+9}$$

(c) requires $y(t) = \mathcal{L}^{-1}\{Y(s)\}$

Note $$Y(s) = \frac{1/s^2}{s^2+9} - e^{-2s} \cdot \frac{1/s^2 + 2/s}{s^2+9}$$

$$= \frac{1}{s^2(s^2+9)} - e^{-2s}\left(\frac{1}{s^2(s^2+9)} + \frac{2}{s(s^2+9)}\right)$$

First treat

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+9)}\right\}$$

$$\frac{A}{s} + \frac{B}{s^2} = \frac{As}{s^2} + \frac{B}{s^2} = \frac{As+B}{s^2}$$

Use partial fractions

mult by C.D. $s^2(s^2+9)$

$$\frac{1}{s^2(s^2+9)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+9}$$

$$1 = As(s^2+9) + B(s^2+9) + (Cs+D)s^2$$

@ $s = 0$

$$1 = 0 + B(0+9) + 0 \implies B = \frac{1}{9}$$

Now

$$1 = As^3 + 9As + \frac{1}{9}s^2 + 1 + Cs^3 + Ds^2$$

(Need to be an identity)

Equate coeffs. of "like terms"

types	LHS		RHS	
const	1	=	1	equal, but not revealing
s	0	=	$9A$	$\implies A = 0$
s^2	0	=	$\frac{1}{9} + D$	$\implies D = -1/9$
s^3	0	=	$A + C$	$\implies C = 0$

Used partial fractions to show

$$\frac{1}{s^2(s^2+9)} = \frac{1/9}{s^2} - \frac{1/9}{s^2+9}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+9)}\right\} = \mathcal{L}^{-1}\left\{\frac{1/9}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1/9}{s^2+9}\right\}$$

$$= \frac{1}{9}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \frac{1}{27}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$$

$$= \underbrace{\frac{1}{9}t - \frac{1}{27}\sin(3t)}$$

Using this

$$\mathcal{L}^{-1}\left\{e^{-2s}\cdot\frac{1}{s^2(s^2+9)}\right\} = \mathcal{U}(t-2)\left(\text{repeated}\ \Big|_{t\mapsto t-2}\right)$$

$$= \mathcal{U}(t-2)\left[\frac{1}{9}(t-2) - \frac{1}{27}\sin(3(t-2))\right]$$

To do $e^{-2s}\cdot\dfrac{2}{s(s^2+9)}$

I'd essentially do all the same work – i.e.

Use partial fractions to split up

$$\frac{2s}{s^2(s^2+9)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+9}$$