Math 251, Wed 6-Oct-2021 -- Wed 6-Oct-2021
Discrete Mathematics
Fall 2021

Wednesday, October 6th 2021

Wk 6, We
Topic:: Big-Oh heirarchy
HW::   PS06 due Thurs.

3. It is a fact that, for all real numbers $x > 2$,

$$10|x^6| \leqslant |17x^6 - 45x^3 + 2x + 8| \leqslant 30|x^6|.$$

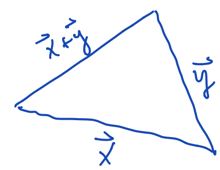
Given this, what sort of Big-O, Big-Ω and/or Big-Θ statements are possible here?

Conclusion: $17x^6 - 45x^3 + 2x + 8$ is $\Theta(x^6)$

(i.e., its "of order x^6").

Triangle Inequality

$|x+y| \leq |x| + |y|$



Some Facts:

1. If $m \geqslant n$ and f is a polynomial of degree n, then $f(x)$ is $O(x^m)$.

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

$$|f(x)| = |a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0| \leq |a_n x^n| + |a_{n-1}x^{n-1}| + \cdots + |a_1 x| + |a_0|$$

$$= |a_n||x^n| + |a_{n-1}||x^{n-1}| + \cdots + |a_1||x| + |a_0|$$

if $x \geq 1$ $\quad \leq |a_n||x^n| + |a_{n-1}||x^n| + \cdots + |a_1||x^n| + |a_0||x^n| = |x^n|(|a_n| + \cdots + |a_1| + |a_0|)$

$\leq |x^m|(|a_n| + \cdots + |a_1| + |a_0|)$ — giving f is $O(x^m)$ w/ witnesses $k = 1$, $C = (|a_0| + |a_1| + \cdots + |a_n|)$.

2. $n!$ is $O(n^n)$ and, as a consequence, $\log_b n!$ is $O(n \log_b n)$, for any $b > 1$.

$$n! = n(n-1)(n-2)\cdots(1) \leq n \cdot n \cdot n \cdots n = n^n$$

$$\log_b n! \leq \log_b(n^n) = n \log_b n$$

3. It can be shown that $n < 2^n$ for $n \geqslant 1$ and, as a consequence, $\log_b n$ is $O(n)$ for all $b > 1$.

$$\log_b n < \log_b(2^n) = n \log_b 2$$

4. If $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$, then $(f_1 + f_2)(x)$ is $O(\max(|g_1(x)|, |g_2(x)|))$.

$$\underset{O(x^2)}{x^2} + \underset{O(x^{1/2})}{3\sqrt{x}} + \underset{O(x)}{\log_2 x} \text{ is } O(x^2).$$

5. If $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$, then $(f_1 f_2)(x)$ is $O(g_1(x) g_2(x))$.

$$x^3(\log_2 x) \text{ is } O(x^4).$$

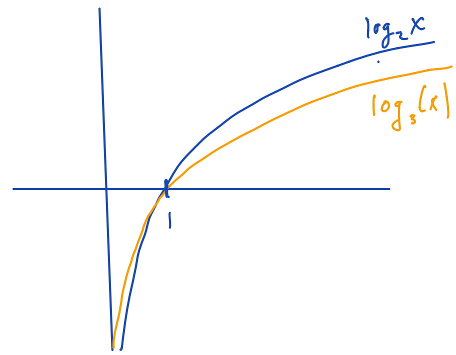
6. As a result of Facts 3 and 5, we have

$$n \log_b n \text{ is } O(n^2), \qquad x^p(\log_b x)^q \text{ is } O(x^{p+q}), \qquad \text{etc.}$$

7. If $f(x)$ is $O(g(x))$ and $g(x)$ is $O(h(x))$, the $f(x)$ is $O(h(x))$.

Transitive property

8. For any values $a, b > 1$, $\log_a x$ is $O(\log_b x)$.



$$\log_a x = \frac{\log_b x}{\log_b a} = \left(\frac{1}{\log_b a}\right) \cdot \log_b x$$

9. Let $c > b > 1$, and $d > 0$. For comparing of a power function x^d with an exponential growth function b^x, we have

$$x^d \text{ is } O(b^x), \qquad \text{but not vice versa.}$$

For comparing the two exponential growth functions c^x, b^x we have

$$b^x \text{ is } O(c^x), \qquad \text{but not vice versa.}$$

So, 2^x is $O(3^x)$, but 3^x is not $O(2^x)$

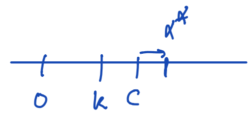
10. It requires calculus, but it can be shown that for any $b > 0$, $c > 0$, $(\log_b x)^c$ is $O(x)$.

There is, therefore, this increasing sequence of orders: 1, $\log_b n$, $(\log_b n)^2$, $(\log_b n)^3$, $\ldots, n$, $n \log_b n$, $n(\log_b n)^2$, $\ldots, n^2$, $n^2 \log_b n$, n^3, $\ldots, 2^n$, 3^n, $\ldots, n!$, n^n.

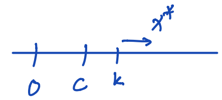
Show that $f(x) = x^2$ is **not** $O(x)$. We'll prove this by contradiction — i.e. assume the opposite is true, and see if lead to a contradiction.

Start: Assume x^2 is $O(x)$, which means there are witnesses $C > 0$ and $k > 0$ such that, whenever $x \geq k$, $C|x| \geq |x^2|$.

Let choose $x^* = 1 + \max(C, k)$. Since $x^* > k$, we have



$$C|x^*| \geq |(x^*)^2| = |x^*||x^*| \geq (C+1)|x^*|$$



Now divide both sides of $\underline{C|x^*|} \geq \underline{(C+1)|x^*|}$ by $|x^*|$

$C > C + 1$ Nonsense!

Theorem 1: Let $f(x)$ be a polynomial of degree n—that is,

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

with $a_n \neq 0$. Then

- $f(x)$ is $O(x^s)$ for all integers $s \geqslant n$.
- $f(x)$ is not $O(x^r)$ for all integers $r < n$.
- $f(x)$ is $\Omega(x^r)$ for all integers $r \leqslant n$.
- $f(x)$ is not $\Omega(x^s)$ for all integers $s > n$.
- $f(x)$ is $\Theta(x^n)$.