Math 251, Wed 13-Oct-2021 -- Wed 13-Oct-2021
Discrete Mathematics
Fall 2021

Wednesday, October 13th 2021

Wk 7, We
Topic:: Induction
Read:: Rosen 5.1
HW:: Moodle Quiz Chs. 2-3 ends Thurs.

$\forall x \in \mathbb{Z}^+, P(x)$

Mathematical Induction

- It is a technique for proving a statement $\forall n \in \mathbb{Z}^+ P(n)$.
- Can be adapted to prove the correctness of some algorithms.
- As a rule of inference, it is

$$(P(1) \wedge \forall k\,(P(k) \to P(k+1))) \to \forall n\, P(n).$$

$P(1)$ is called the **basis step**, $P(k) \to P(k+1)$ is called the **inductive step**, and the assumption that the hypothesis $P(k)$ of the inductive step holds is called the **inductive hypothesis**.

Induction is not helpful in discovering new mathematical statements which are true. Once a pattern or truth has been conjectured, however, induction can often establish that it is true.

Examples:

$P(n)$ says "the sum of the first n positive odd integers is n^2."

1. $\displaystyle\sum_{j=1}^{n}(2j-1) = 1+3+5+\cdots+(2n-1) = n^2$

basis step: $P(1)$ s s $1 = 1^2$

induction step:

Can assume $P(k)$ holds – i.e., that $1+3+\cdots+(2k-1) = k^2$ for some $k \geq 1$. Must show: $1+3+\cdots+(2k-1)+(2k+1) = (k+1)^2$

See below

2. For all positive integers, $23^n - 1$ is divisible by 11.

Here $P(n)$: " $11 \mid 23^n - 1$ "

basis step: $P(1)$ says "$23-1$ is divisible by 11"

induction step: See below

3. For all positive integers, $n < 2^n$.

Here $P(n)$: $n < 2^n$. basis step: $P(1)$ says $1 < 2$.

induction step: Can assume $P(k)$ holds for some k, that is $k < 2^k$.

must show: $k+1 < 2^{k+1}$.

$$k+1 < 2^k + 1 < 2^k + 2^k = 2(2^k) = 2^{k+1}.$$

by induction hypothesis

by induction hypothesis

$$\underbrace{1+3+5+\cdots+(2k-1)}_{\text{first } k \text{ pos. odds}} + \underbrace{(2k+1)}_{(k+1)^{st} \text{ pos. odd}} = k^2 + (2k+1)$$

$$= (k+1)^2$$

#2 induction step: $P(k) \longrightarrow P(k+1)$ says $23^{k+1}-1$ divisibly by 11

Assume $P(k)$ holds for some $k \in \mathbb{Z}^+$ — i.e.,

$23^k - 1$ is divisible by 11.

$$23^{k+1}-1 = 23^{k+1} - 23^k + 23^k - 1$$

$$= (23^{k+1}-23^k) + (23^k - 1)$$

$$= 23^k(23-1) + (23^k-1)$$

$$= \underbrace{23^k(22)}_{\text{"clearly" divisible by 11}} + \underbrace{(23^k-1)}_{\text{divisible by 11 by induction hyp.}}$$

which gives a sum that is divisible by 11.

By induction, $\forall n \in \mathbb{Z}^+$, $P(n)$ holds.

4. For all $n \in \mathbb{N} - \{0, 1, 2, 3\}$, $2^n < n!$

$P(n)$: $2^n < n!$

basis step: $P(4)$ says $2^4 < 4!$

induction step: Assume $P(k)$, i.e., $2^k < k!$, holds for some $k \geq 4$.

Need to show $P(k+1)$ holds — i.e. $2^{k+1} < (k+1)!$

$$2^{k+1} = 2 \cdot 2^k \underset{\substack{\uparrow \\ \text{inductive} \\ \text{hypothesis}}}{<} 2k! < (k+1) \cdot k! = (k+1)!$$

5. If B is a set with $|B| = n$, then $|\mathcal{P}(B)| = 2^n$, for all $n \in \mathbb{N}$.