Math 251, Mon 18-Oct-2021 -- Mon 18-Oct-2021
Discrete Mathematics
Fall 2021

Monday, October 18th 2021

Due:: WW MathematicalInduction due Mon.

Monday, October 18th 2021

Wk 8, Mo
Topic:: Induction
Topic:: Strong induction
Read:: Rosen 5.2

2 is prime
3 is prime
$4 = 2 \cdot 2$ composite
5 is prime
$6 = 2 \cdot 3$ composite
$24 = 2 \cdot 2 \cdot 2 \cdot 3$

Strong Induction and the Well-Ordering Principle

Mathematical induction can be expressed as the rule of inference

$$(P(1) \wedge \forall k \in \mathbb{Z}^+, (P(k) \to P(k+1))) \to \forall n \in \mathbb{Z}^+, P(n).$$

Example 1: Fundamental Theorem of Arithmetic

Every integer $n \geqslant 2$ is a prime or can be written as the product of primes (~~use strong mathematical induction~~).

$P(k)$: k is prime or the product of primes.

Note: A **prime** number is a positive integer greater than 1 that has no positive (integer) factors besides 1 and itself.

Basis step: $P(2)$ says "2 is prime".

Usual Induction step: Assume $P(k)$ holds for some $k \geq 2$. Show $P(k+1)$.

$P(k+1)$ says $k+1$ is prime or prod. of primes.

If the $k+1$ is not prime, then

$k+1 = pq$ with $1 < p, q \leq k$

Induction, as described above, is inadequate.

Consider this modified version:

$$(P(1) \wedge \forall k \in \mathbb{Z}^+, (\underline{P(1) \wedge P(2) \wedge \cdots \wedge P(k)} \to P(k+1))) \to \forall n \in \mathbb{Z}^+, P(n).$$

As before, if $k+1$ is not prime and

$k+1 = pq$ w/ $1 < p, q \leq k$

then this (new) induction hypothesis says

$p = p_1 p_2 \cdots p_\ell$ (primes generating p)

$q = q_1 q_2 \cdots q_m$ (primes generating q)

$\to k+1 = (p_1 p_2 \cdots p_\ell) \cdot (q_1 q_2 \cdots q_m)$ showing $P(k+1)$ holds.

■

Example 2:

For any $n \geqslant 8$, n cents can be obtained using 3¢ and 5¢ coins.

$8 = 3 + 5$
$9 = 3(3)$
$10 = 2(5)$
$11 = 2(3) + 5 = 8 + 3$

Induction step: If $P(8) \wedge P(9) \wedge P(10) \wedge \cdots \wedge P(k)$ hold (IH)

Need $k \geq 10$

then, in particular, $P(k-2)$ holds — that is

$$k - 2 = l(3) + m(5)$$

and therefore

$$k+1 = (k-2) + 3 = l(3) + m(5) + 3$$
$$= (l+1)(3) + m(5).$$

Basis Step: $P(8)$, $P(9)$, $P(10)$ done here

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A generalization (equally valid form) of mathematical induction.

Definition 1 (Principle of Strong Mathematical Induction): Let $P(n)$ be a property that is defined for integers n, and let a, b be fixed integers with $a \leqslant b$. Suppose the following statements are true:

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1. $P(a), P(a+1), \ldots, P(b)$ are all true (**basis step**).
2. For any integer $k \geqslant b$, if $P(i)$ is true for all integers i from a through k, then $P(k+1)$ is true (**inductive step**). $P(a) \wedge P(a+1) \wedge P(a+2) \wedge \cdots \wedge P(k)$

Then the statement "for all integers $n \geqslant a$, $P(n)$" is true. $\rightarrow P(k+1)$

The supposition that $P(i)$ is true for all integers i from a through k in number 2 above is designated as the **inductive hypothesis** in this form of induction.

We have two equivalent notions of induction. A third notion that looks different but is nonetheless equivalent is the **Well-Ordering principle**.

with $A \neq \{\}$

Definition 2 (Well-Ordering Principle): Suppose $A \subseteq \mathbb{N}$. Then A has a *smallest element*. That is, $\exists a \in A$ such that $\forall b \in A$, $(a \leqslant b)$.

Note that the set {positive real numbers} does not have a smallest element, but that this does not violate the well-ordering principle.

1. Let $a_0, a_1, a_2, \ldots$ be the sequence defined by the 2^{nd}-order linear recursion relation

$$a_n = 6a_{n-1} - 5a_{n-2}, \quad \text{for } n \geqslant 2, \quad \text{with } a_0 = 0,\ a_1 = 4.$$

Take $P(n)$: $a_n = 5^n - 1$. Then $\forall n \in \mathbb{N}$, $P(n)$ (use strong mathematical induction).

Basis step: $P(0)$ says $a_0 = 5^0 - 1 = 1 - 1 = 0$

$P(1)$ says $a_1 = 5^1 - 1 = 5 - 1 = 4$

Induction Step: Now assume $P(0), P(1), \ldots, P(k)$ all hold for $k \geq 1$.

So $k+1 \geq 2$, so

shows $P(k+1)$ holds.

$$a_{k+1} = 6a_k - 5a_{k-1} \overset{\text{by IH}}{=} 6(5^k - 1) - 5(5^{k-1} - 1)$$
$$= 6(5^k) - 6 - 5^k + 5 = 5^k(6-1) - 1$$
$$= 5^k \cdot 5 - 1 = 5^{k+1} - 1.$$

2. Use strong mathematical induction to show the product of n numbers requires $n-1$ multiplications, regardless of grouping.

$P(n)$: If you wish to multiply n numbers, you'll need $n-1$ multiplications (flops)

Basis step: $P(1)$ says you don't need any multiplications.

Induction step: Assume we have $k+1$ numbers, w/ $k \geq 1$.

Inductive hyp: $P(1), P(2), \ldots, P(k)$ hold.

$$\underbrace{(a_1 \cdot a_2 \cdots a_m)}_{\text{contains } m \text{ nos. w/ } m \geq 1} \cdot \underbrace{(a_{m+1} \cdots a_{k+1})}_{\text{contains } k-m+1 \text{ nos.}}$$

Note $k-m+1 \leq k$

requires $m-1$ multiplies, by IH

requires $k-m$ multiplies by IH

$\longrightarrow (m-1) + (k-m) + 1$
$= k$ multiplies