David, Oscar

Test 2 Wed. Oct. 27
Coverage:
Chap 2: Sections 4-5
Chap 3: Sections 1-3
Chap 5: Sections 1-2

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Math 251, Mon 25-Oct-2021 -- Mon 25-Oct-2021
Discrete Mathematics
Fall 2021

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Monday, October 25th 2021
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Wk 9, Mo
Topic:: Time spent on HW problems
[[ notes/lect28Oct19.pdf
Read:: Rosen 5.3
HW:: Moodle Quiz Chs. 5 ends Wed.
HW[[   PS12 due Fri.
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Supplied Formulas for Test 2

For an arithmetic sequence, $a_0, a_1, a_2, \ldots,$ the sum of the first $n+1$ terms

$$a_0 + a_1 + \cdots + a_n \;=\; (n+1)\frac{a_0 + a_n}{2}.$$

For a geometric sequence, $a_0, a_0 r, a_0 r^2, \ldots,$ the sum of the first $n+1$ terms

$$a_0 + a_0 r + \cdots + a_0 r^n \;=\; a_0\,\frac{1 - r^{n+1}}{1 - r}.$$

The sum of all infinitely-many terms exists only when $|r| < 1$, in which case it is

$$a_0 + a_0 r + \cdots + a_0 r^n + \cdots \;=\; a_0\,\frac{1}{1 - r}.$$

If you think of others, feel free to ask.
The default answer: No.

Test: Ch. 2 Sections 2.4-2.5
Ch. 3 Sections 3.1-3.3
Ch. 5 Sections 5.1-5.3

Problem examples
- from these textbook sections
- WebWork
- Quiz questions

5.2.2

$P(n)$ - the n^{th} domino falls

Show: $\forall n \in \mathbb{Z}^+ \; P(n)$

In problem statement you learn $P(1), P(2), P(3)$ hold.

also $P(k) \to P(k+3)$.

k

✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, . . .

Actual argument

basis step: $P(1), P(2), P(3)$ hold (given)

strong induction step: Assume, for some $k \geq 3$

$P(1) \wedge P(2) \wedge \cdots \wedge P(k)$ hold.

Now to show $P(k+1)$, note that $P(k-2)$ holds since $k \geq 3$; by the induction hypothesis, and problem statement tells us

$$P(n) \to P(n+3).$$

Thus, $P(k-2) \to P(k+1)$. So $\forall n, P(n)$.

Prior example

3¢, 5¢ coins

$P(n)$: Can make n cents as combination of $m(3) + l(5)$.

$\forall n \in \{8, 9, 10, \ldots\}, \; P(n)$.

Argument

basis step: $P(8) = 5 + 3$

$P(9) = 3(3)$

$P(10) = 2(5)$

Induction hypothesis (IH)

Strong Induction step: Suppose for some $k \geq 10$, $P(8) \wedge P(9) \wedge \cdots \wedge P(k)$ holds.

Show $P(k+1)$

We have $P(k-2)$ in our IH.

So $k-2 = m(3) + \ell(5)$

Thus $k+1 = 3 + m(3) + \ell(5)$

$= (m+1)(3) + \ell(5)$, proving $P(k+1)$ holds.

Math induction

$$\underset{\text{base case}}{\underset{\uparrow}{P(a)}} \wedge \forall k, \big(P(k) \to P(k+1)\big) \longrightarrow \forall k \geq a, P(k)$$

Strong induction ladder rung a

$$P(a) \wedge \forall k \big(P(a) \wedge P(a+1) \wedge \dots \wedge P(k)\big) \longrightarrow \forall k \geq a, P(k)$$

Example from: $P(n)$ says n is prime or the product of primes

Prove "$\forall n \geq 2, P(n)$"

basis step: $P(2)$

induction step: Assume k integer w/ $k \geq 2$ and

$$\underbrace{P(2) \wedge P(3) \wedge \dots \wedge P(k)}_{\text{may only be } P(2).}$$

Now show $P(k+1)$:

$k+1$ may be a prime. If so, $P(k+1)$ holds.

But if it isn't, $k+1 = pq$ w/ p, q both in $\{2, \dots, k\}$.

$P(p)$ and $P(q)$ hold —

$$\left.\begin{aligned} p &= p_1 p_2 \cdots p_m \\ q &= q_1 q_2 \cdots q_l \end{aligned}\right\} \longrightarrow \quad k+1 = (p_1 \cdots p_m)(q_1 \cdots q_l).$$

$O(1)$ before $O(\log x)$

" $O((\log x)^2)$

" $O((\log x)^3)$

$\vdots$

" $O(x)$

" $O(x \log x)$

" $O(x (\log x)^2)$

$\vdots$

" $O(x^2)$

$x^2 + 2x^3 \log x - (\log x)^{10}$

is $O(x^3 \log x)$

$O(x^4)$

2^x

2.1^x

3^x

$\vdots$

Prove: $x^2 + 3x$ is $\Theta(x^2)$

Requires 1) x^2+3x is $O(x^2)$

2) x^2+3x is $\Omega(x^2)$ or alternatively, show x^2 is $O(x^2+3x)$

Prove: x^2 is not $O(x)$.

Do by contradiction.

Assume the opposite, namely that x^2 is $O(x)$.

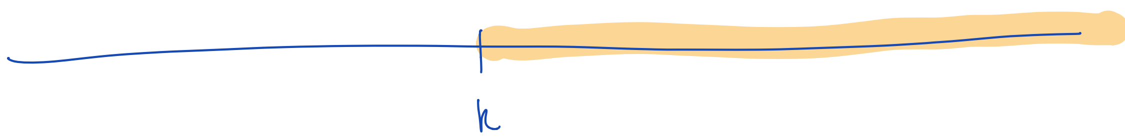
Then, by definition, this means there are witnesses $C, k > 0$ such that

$$|x^2| \le C|x| \quad \text{for } x \ge k.$$

When one has witnesses in hand, it is possible to replace either C or k with an even higher number to get another valid pair of witnesses. Based on that fact, assume that $C > k$.

Abs. vals unnecessary for $x > k > 0$.

$$x^2 \le Cx \longrightarrow x \le C$$



So, the contradiction can be summarized like this:

If we take $x_0 = C+1$ which is a number satisfying $x_0 > k$, it should be that

$$x_0^2 \le C x_0, \quad \text{or} \quad (C+1)^2 \le C(C+1).$$

But that is not so, since it implies $C+1 < C$, which is not so.

What led to this contradiction is the original assumption that x^2 is $O(x)$, which must then be false.

12:30 Section

Ex. of Big Oh question:

Easy: Is $x^2 - x(\log x)^2 + x^2 \log x$ $O(x^2)$? No

Harder: Show $3x^2 + 2x + 7$ is $\Theta(x^2)$.

Do it from the definition – produce witnesses.

1. Need $3x^2 + 2x + 7$ is $O(x^2)$

2. Show $3x^2 + 2x + 7$ is $\Omega(x^2)$

(or, show x^2 is $O(3x^2 + 2x + 7)$).

1st Task:

$$|3x^2 + 2x + 7| \underset{x>0}{=} 3x^2 + 2x + 7 \underset{x \geq 1}{\leq} 3x^2 + 2x^2 + 7x^2 = 12x^2$$

Witnesses $k = 1$, $C = 12$, Task 1 complete,

(for task 1)

2nd task:

To get x^2 is $O(3x^2 + 2x + 7)$, note

$$|x^2| \underset{x>0}{=} x^2 \leq (3x^2 + 2x + 7) \cdot 1$$

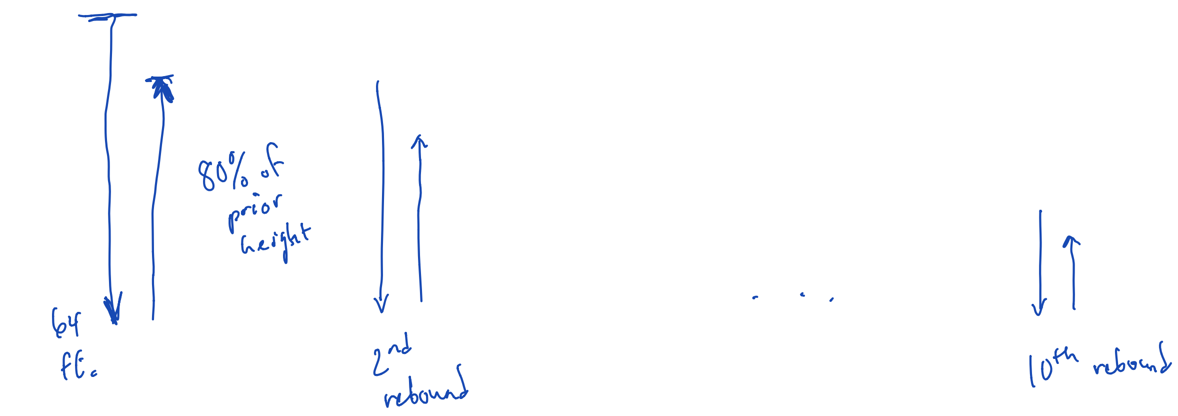
Witnesses: $k = 0$, $C = 1$ (for task 2)

So, witnesses to Θ relationship

$$C_1 |x^2| \leq |3x^2 + 2x + 7| \leq C_2 |x^2| \quad \text{for } x \geq k$$

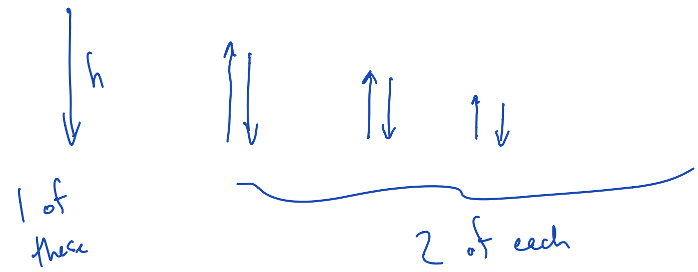
are $C_1 = \frac{1}{1}$, $C_2 = 12$, $k = 1$

Rebounding balls



How Moodle quiz problem was stated

"Downs": $64 + (64)(0.8) + 64(0.8)^2 + \cdots + 64(0.8)^9$



Book problem included <u>all</u> (infinitely many) rebounds

Falls originally h

$$+ 2hr$$

$$+ 2hr^2$$

$$+ 2hr^3$$

$$+ \cdots$$

both geometric

$$= \underbrace{\left(h + hr + hr^2 + \cdots \right)}_{\text{downs}} + \underbrace{\left(hr + hr^2 + \cdots \right)}_{\text{ups}}$$

$$= h\left(1 + r + r^2 + \cdots\right) + hr\left(1 + r + r^2 + \cdots\right)$$

$$= \frac{1}{1-r}$$

$$= h \cdot \frac{1}{1-r} + hr \cdot \frac{1}{1-r}$$

$$a_0 + a_0 r + a_0 r^2 + \cdots + a_0 r^{n-1} = \left(\frac{1-r^n}{1-r}\right) a_0$$

5.2.8

$25 = 1(25) + 0(40)$
$40 = 0(25) + 1(40)$
$50 = 2(25) + 0(40)$
$65 = 1(25) + 1(40)$
$75 = 3(25)$
$80 = 2(40)$
$90 = 2(25) + 1(40)$

$140 = 4(25) + 40$
$145 = 1(25) + 3(40)$
$150 = 6(25)$
$155 = 3(25) + 2(40)$
$160 = 4(40)$

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$P(n)$: Can produce $5n$ dollars using \$25, \$40 denominations

We can prove: $\forall n \in \{28, 29, \ldots\},\ P(n)$.

basis step: $P(28), P(29), P(30), P(31), P(32)$

induction step: Assume (for strong induction) that, for some $k \geq 32$,
$P(28) \wedge P(29) \wedge \cdots \wedge P(k)$ holds

To show $P(k+1)$, note that $P(k-4)$ is assumed in my induction hypothesis (since $k \geq 32$). So

$5(k-4) =$ (some number of \$25) + (some no. of \$40)

Thus $5(k+1) =$ (augment w/ one more \$25) + (same no. of \$40) proves $P(k+1)$.