Math 251, Fri 29-Oct-2021 -- Fri 29-Oct-2021
Discrete Mathematics
Fall 2021

Friday, October 29th 2021

Wk 9, Fr
Topic:: Structural induction
Read:: Rosen 5.3
HW:: PS08 due Tues.

base cases
recursive step

Structural Induction

Suppose, previously, a recursive definition R has been used to define objects.
Let S be a statement about elements defined by R.

If
(i) S is True for every element $b_1, \ldots, b_m$ in the base case of definition R, and
(ii) whenever elements $e_1, \ldots, e_n$, about which the statement S is true, are used to construct a new element E via the recursive step in R, statement S also holds for this constructed element E (that is, if S holds for $e_1, \ldots, e_n$ then S holds for E).

then we conclude S is true for all elements built using recursive definition R.

Observe:

$P(k) \to P(k+1)$ $P(a) \wedge P(a+1) \wedge \cdots \wedge P(k) \to P(k+1)$

- Mathematical and strong induction require predicate statements which change with integer inputs. Structural induction carries the key idea of previous induction types to things like lists, strings, trees, ..., data structures.
- the basis step and induction steps in structural induction mirror the base cases and recursive step in the corresponding recursive definition.

Examples

1. Consider this recursive definition R:

   base case: the empty expression λ

   recursive step:
   - Given an already-admitted expression S, admit (S) ( ) (( ))
   - Given two already-admitted expressions S_1, S_2, admit S_1S_2)(( )( ) ~~( )( )~~

   ((( )))( )

   For each S defined through R, let

   $$c(S) = (\text{number of left parentheses in S}) - (\text{number of right parentheses in S})$$

   Claim: $\forall S, c(S) = 0$.

(i) show $c(\lambda) = 0$ (self-evident, $0 - 0 = 0$).

(ii) To take care of first bullet point

Given $C(S) = 0$, need to show (unless self-evident) $C((S)) = 0$. ✓

To take care of 2nd bullet

Given that $C(S_1) = 0$ and $C(S_2) = 0$,
show (unless self-evident) $C(S_1 S_2) = 0$.

2. Consider this recursive definition R defining a class of strings:

   base case:

   - admit empty string λ
   - admit the string aab

   recursive step: Given already-admitted strings $w_1, w_2, \ldots, w_n$, admit the concatenated string $w_1 w_2 \cdots w_n$

   aabaab

   aabaabaabaabaab

   Suggest and prove some claims using structural induction.

1. length of any string is divisible by 3.
   (i) true for λ, aab
   (ii) Every one of $w_1, w_2, \ldots, w_n$ has length divisible by 3 $\xrightarrow{\text{self-evident}}$ so does $w_1 w_2 \ldots w_n$.
2. # of a's in a string = 2 (# of b's).
3. No word with side-by-side b characters.

3. Consider a different collection of strings, defined by

   base case: admit the string a

   recursive step: Given an already-admitted string S, admit aSb.

   aab

   aaabb

   ~~aaaaabb~~

   Suggest and prove some claims using structural induction.

Claim: # of b's in a string = (# of a's in string) − 1

4. Consider this recursive construction of polynomials.

 base case: $1, x$

 recursive step:

 - Given an already-admitted polynomials p, q, admit $p + q, pq$
 - Given an already-admitted polynomial p, and $c \in \mathbb{R}$, admit cp

 Claim: Every power of x is a nonnegative integer

$1 + x$

$\pi x, \; 17, \; -3$

$3x^1 - 2x^3 + 17x^0$

5. Define a doubly-subscripted sequence $a_{m,n}$ recursively this way.

 base case: $a_{0,0} = 0$

 recursive step:

 - For $m \geqslant 1$, let $a_{m,0} = 1 + a_{m-1,0}$ — building block for this
 - For $n \geqslant 1$, let $a_{m,n} = n + a_{m,n-1}$

 Claim: $a_{m,n} = m + n(n+1)/2$

$m, n \in \mathbb{N}$

$a_{m,n}$

$a_{m,n-1}$

$a_{0,3} \quad a_{1,3}$

$a_{0,2} \quad a_{1,2}$

$a_{0,1} \quad a_{1,1}$

$a_{0,0} = 0 \quad a_{1,0} \quad a_{2,0} \quad a_{3,0}$

(i) Does $a_{0,0}$ (which is given to be 0) equal $0 + 0(0+1)/2$?

(ii) Need to handle both bullet points.

For the first, can assume $a_{m-1,0} = m-1 + 0(0+1)/2 = m-1$.

Need to show that the new term $a_{m,0}$ from $a_{m-1,0}$ also fits this.

Know: $a_{m,0} = 1 + a_{m-1,0} = 1 + (m-1) = m$

$\stackrel{?}{=} m + 0(0+1)/2 \quad \checkmark$

For the second bullet,

Can assume $a_{m,n-1}$ is found according to my claimed formula

$$a_{m,n-1} = m + (n-1)(n-1+1)/2 = m + \frac{n(n-1)}{2}$$

Must show $a_{m,n}$ fits the formula.

Know

$$\begin{aligned} a_{m,n} &= n + a_{m,n-1} \\ &= n + m + \frac{n(n-1)}{2} \\ &= m + n + \frac{1}{2}n^2 - \frac{1}{2}n \\ &= m + \frac{1}{2}n^2 + \frac{1}{2}n = m + \frac{n^2+n}{2} \\ &= m + \frac{n(n+1)}{2}. \quad \checkmark \end{aligned}$$