Math 251, Mon 1-Nov-2021 -- Mon 1-Nov-2021

Discrete Mathematics

Fall 2021

Monday, November 1st 2021

Wk 10, Mo

Topic:: Solving linear homogeneous recurrences

Read:: Rosen 8.2A

Chs. 8

8.1 Motivation

8.2 Solution methods

8.3 Master Theorem

Ch. 4

4.1–4.6

Linear homogeneous recurrences

specific constants

A linear homogeneous recurrence relation of degree k with constant coefficients has the form

$$\boxed{a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}.}$$

Examples and non-examples:

extra 'd' term is a nonhomogeneous term

1. $a_n = a_{n-1} + d$ not a fit to our form since d-term present
2. $a_n = r a_{n-1}$ is linear, homogeneous, degree 1
3. $f_n = f_{n-1} + f_{n-2}$ is linear, homog., degree 2
4. $w_n = 2w_{n-1} + w_{n-5}$ is linear, homog., degree 5
5. $C_n = C_0 C_{n-1} + C_1 C_{n-2} + \cdots + C_{n-1} C_0$ is not linear (multiplications occur between unknowns)
6. $a_n = 2a_{n-1} + 2a_{n-2} + 3^{n-2}$ is linear, non-homogeneous

nonhomog. term

Solution method:

- Assume each member $a_0, a_1, \ldots, a_n$ of the sequence that solves has the form r^k—that is, $a_0 = r^0, a_1 = r, \ldots, a_n = r^n$, and so on. Substitute these values into the recurrence.
- Solve the resulting **characteristic equation** for roots r. For a degree k recurrence, there will be k roots.
- Take an appropriate weighted combination of solutions, and use initial conditions to determine the weights.

Examples:

1. Solve $a_n = a_{n-1} + 2a_{n-2}$, $a_0 = 1$, $a_1 = 8$
2. Solve $f_n = f_{n-1} + f_{n-2}$, $f_0 = 0$, $f_1 = 1$

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Facts (focus on 2nd degree problems when stating):
 a) Eqn (**) has k=2 roots (counting multiplicities, complex number roots)
 b) For each root r=r_{1,2}, the sequence
      a_n = r_j^n  satisfies the recurrence relation (*)
 c) linear combinations of solutions to (*) again solve (*)
      a_n = alpha_1 r_1^n + alpha_2 r_2^n
 d) if the roots of (**) are distinct, then all solutions of (*) look like
      a_n = alpha_1 r_1^n + alpha_2 r_2^n
    if there is a repeated root, the solutions look like
      a_n = (alpha_1 + alpha_2 n) r^n
 e) If our recurrence relation comes with a full contingent of ICs, then
    there is one and only one choice of alpha_j's so that these are also met.
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$$\overbrace{}^{\text{initial conditions}}$$

1. $a_n = a_{n-1} + 2a_{n-2}$, $\quad a_0 = 1, \quad a_1 = 8$

- Given $a_n = r^n$, insert into

  $a_n = a_{n-1} + 2a_{n-2}$ so it becomes $r^n = r^{n-1} + 2r^{n-2}$

  Algebraically, this can be rearranged

  $r^n - r^{n-1} - 2r^{n-2} = 0$ or $r^{n-2}(r^2 - r - 2) = 0$

  Since their product is 0, either

  $$\underbrace{r^{n-2} = 0}_{\text{uninteresting, not useful}} \quad \text{or} \quad \underbrace{r^2 - r - 2 = 0}_{\text{important - call it "characteristic eqn"}}$$

- $r^2 - r - 2 = 0$

  by factoring get $(r-2)(r+1) = 0 \longrightarrow$ roots $r = -1, 2$

  Note: characteristic polynomial $r^2 - r - 2$ is degree 2 and we wind up w/ 2 roots.

  Each root generates, through assumption $a_n = r^n$, a sequence

	r^0	r^1	r^2	r^3	r^4	...
$r = -1$	1	-1	1	-1	1	...
$r = 2$	1	2	4	8	16	...

  and both sequences satisfy the recurrence relation.

- $a_n = \alpha_1(-1)^n + \alpha_2(2)^n$

  α_1, α_2 are weights as-yet-unknown – use initial conditions to find them

  ICs

  $a_0 = 1$ but weighted formula says $a_0 = \alpha_1(-1)^0 + \alpha_2(2)^0$

  Equating these, $1 = \alpha_1 + \alpha_2$

$a_1 = 8$, but also $a_1 = \alpha_1(-1)^1 + \alpha_2(2)^1$

Equating: $-\alpha_1 + 2\alpha_2 = 8$

Have 2 eqns in unknowns α_1, α_2

$$\begin{array}{rcl} \alpha_1 + \alpha_2 &=& 1 \\ -\alpha_1 + 2\alpha_2 &=& 8 \\ \hline 3\alpha_2 &=& 9 \end{array} \longrightarrow \alpha_2 = 3$$

$$\alpha_1 = -2$$

Now we have arrived at a soln. to both

the recurrence, and

ICs

$$a_n = -2(-1)^n + 3(2)^n.$$

2. $f_n = f_{n-1} + f_{n-2}$, $f_0 = 0$, $f_1 = 1$

- Assume $f_n = r^n$, $n = 0, 1, 2, \ldots$, then substitute into recurrence

$$r^n = r^{n-1} + r^{n-2} \longrightarrow r^n - r^{n-1} - r^{n-2} = 0$$

$$\longrightarrow r^{n-2}\underbrace{(r^2 - r - 1)}_{\text{char. polynomial}} = 0$$

- Solve char. eqn

$$r^2 - r - 1 = 0$$

Quadratic formula

Solns. to $ax^2 + bx + c = 0$

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

Applying quad. formula

$$r = \frac{1}{2} \pm \frac{\sqrt{(-1)^2 - 4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}.$$

	r^0	r^1	r^2	r^3	$\ldots$	
$\frac{1+\sqrt{5}}{2}$:	1	1.618	2.618	4.236	6.854	$\ldots$
$\frac{1-\sqrt{5}}{2}$:	1	−0.618	0.382	−0.2361	0.1459	$\ldots$

Both satisfy the recurrence relation.

Neither looks like the Fibonacci sequence.