Try same process w/ Fibonacci recurrence

$$f_n = f_{n-1} + f_{n-2}, \quad f_0 = 0, \quad f_1 = 1.$$

Assume $f_n = r^n$ and substitute

$ax^2 + bx + c = 0$

has solns.

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$r^n = r^{n-1} + r^{n-2} \implies r^n - r^{n-1} - r^{n-2} = 0$$

$$r^{n-2}(r^2 - r - 1) = 0$$

Char. eqn. $r^2 - r - 1 = 0$.

$$r = \frac{1}{2(1)} \pm \frac{\sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \frac{1}{2} \pm \frac{\sqrt{5}}{2}$$

$$r_1 = \frac{1+\sqrt{5}}{2}, \quad r_2 = \frac{1-\sqrt{5}}{2}$$

Began here today

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	r^0	r^1	r^2	r^3	r^4	
$r_1 \approx 1.618$	1	1.618	2.618	4.236	6.854	...
$r_2 \approx -0.618$	1	-0.618	0.382	-0.236	0.146	...

both solve recurrence, neither gives correct initial values.

$$f_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$

hope to choose weights α_1, α_2 to give formula for f_n

Determine weights using initial values

formula	ICs	
$f_0 = \alpha_1 r_1^0 + \alpha_2 r_2^0 = \alpha_1 + \alpha_2$	$f_0 = 0$	$\rightarrow \alpha_1 + \alpha_2 = 0$
$f_1 = \alpha_1 r_1^1 + \alpha_2 r_2^1 = \alpha_1\left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right)$	$f_1 = 1$	

$$1 = \alpha_1 \cdot \frac{1}{2} + \alpha_1 \cdot \frac{\sqrt{5}}{2} + \alpha_2 \frac{1}{2} - \alpha_2 \frac{\sqrt{5}}{2}$$

2 eqns.

1) $\alpha_1 + \alpha_2 = 0 \longrightarrow \alpha_1 = -\alpha_2$

substitute for α_1

2) $\alpha_1 \cdot \frac{1}{2} + \alpha_1 \frac{\sqrt{5}}{2} + \alpha_2 \cdot \frac{1}{2} - \alpha_2 \frac{\sqrt{5}}{2} = 1$

$$\cancel{-\alpha_2 \frac{1}{2}} - \alpha_2 \frac{\sqrt{5}}{2} + \cancel{\alpha_2 \frac{1}{2}} - \alpha_2 \frac{\sqrt{5}}{2} = 1$$

$$\rightarrow \quad -\alpha_2 \sqrt{5} = 1$$

$$\rightarrow \quad \alpha_2 = \frac{-1}{\sqrt{5}}, \quad \alpha_1 = \frac{1}{\sqrt{5}}$$

Had proposed:

$$f_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$

$$= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Example: $a_n = 3a_{n-1} + 2, \quad a_0 = 5$

1^{st}-degree nonhomog. linear recurrence
Can't directly be solved by assuming $a_n = r^n$.

Can solve iteratively (not same process as 8.2; encountered in 2.4)

$$a_n = 3a_{n-1} + 2 = 3(3a_{n-2} + 2) + 2 = 3^2 a_{n-2} + 3 \cdot 2 + 2$$

$$= 3^2(3a_{n-3} + 2) + 3 \cdot 2 + 2 = 3^3 a_{n-3} + 3^2 \cdot 2 + 3 \cdot 2 + 2$$

$$= 3^3(3a_{n-4} + 2) + 3 \cdot 2^2 + 3 \cdot 2 + 2 = 3^4 a_{n-4} + \underline{3^3 \cdot 2 + 3^2 \cdot 2 + 3 \cdot 2 + 2}$$

$$= \ldots$$

$$= 3^n a_0 + 3^{n-1} \cdot 2 + \ldots + 3^3 \cdot 2 + 3^2 \cdot 2 + 3 \cdot 2 + 2$$

$$= 3^n a_0 + 2(\underbrace{1 + 3 + 3^2 + \ldots + 3^{n-1}}_{\text{first } n \text{ terms of a geometric sequence}}) \qquad = \frac{1-3^n}{1-3}$$

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More Examples:
 3. Solve b_n = 4b_{n-1} - 4b_{n-2}, with ICs b_0 = 1, b_1 = 3
      answer: b_n = 2^n(1 + n/2)
 4. What sorts of solutions to
      a_n = 4a_{n-1} + 11 a_{n-2} - 30 a_{n-3}?   (roots are -3, 2, 5)
      a_n = 3a_{n-1} - 3 a_{n-2} + a_{n-3}?   (roots are 1, 1, 1)
      a_n = 7a_{n-1} - 16 a_{n-2} + 12 a_{n-3}?   (roots are 2, 2, 3)
 5. Suppose one has a 6th degree linear homog. RR with CC's, and the
    characteristic poly. has roots 1, 3, 3, 3, 6, 6
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Example 3: $b_n = 4b_{n-1} - 4b_{n-2}$, $b_0 = 1$, $b_1 = 3$

Start w/ $b_n = r^n$ (so $b_{n-1} = r^{n-1}$, $b_{n-2} = r^{n-2}$)

Insert into recurrence

$$r^n = 4r^{n-1} - 4r^{n-2}, \quad \text{or} \quad r^n - 4r^{n-1} + 4r^{n-2} = 0$$

$$\text{or} \quad r^{n-2}(r^2 - 4r + 4) = 0$$

Solve $r^2 - 4r + 4 = 0$

$(r-2)(r-2) = 0 \rightarrow$ roots $r_{1,2} = 2$.

Repeated root $r = 2$

	2^0	2^1	2^2	2^3	2^4	
2	1	2	4	8	16	...

2^{nd} root? $\rightarrow$ 2 __ __ __ __ __ (match top row)

Important fact:

When r is a repeated root of the char. poly., then not only does r^n satisfy the recurrence, so does $n \cdot r^n$.

In our setting, 2 was a double root, so

$$2^n: \quad 1, \; 2, \; 4, \; 8, \; 16, \; \ldots$$

solves the recurrence eqn,

$$2^0 \quad 2^1 \quad 2^2 \quad 2^3 \quad 2^4$$

but so does

$$n \cdot 2^n: \quad 0, \; 2, \; 8, \; 24, \; 64, \; \ldots$$

solves, too.

$$0 \cdot 2^0 \quad 1 \cdot 2^1 \quad 2 \cdot 2^2 \quad 3 \cdot 2^3 \quad 4 \cdot 2^4$$

Use these two as before, taking a weighted sum

$$b_n = \alpha_1 \cdot 2^n + \alpha_2 \cdot n \cdot 2^n$$

Use ICs:

$$1 = b_0 = \alpha_1 \cdot 2^0 + \alpha_2 \cdot 0 \cdot 2^0 = \alpha_1$$

$$3 = b_1 = \alpha_1 \cdot 2^1 + \alpha_2 \cdot 1 \cdot 2^1 = 2\alpha_1 + 2\alpha_2$$

$$3 = 2(1) + 2\alpha_2 \longrightarrow \alpha_2 = \frac{1}{2}$$

So our solution formula

$$b_n = 1 \cdot 2^n + \frac{1}{2} \cdot n \cdot 2^n.$$

4. roots are $-3, 2, 5$ so expect

$$a_n = \alpha_1 (-3)^n + \alpha_2 2^n + \alpha_3 5^n$$

roots are $4, 4, 4$

$$a_n = \alpha_1 4^n + \alpha_2 n \cdot 4^n + \alpha_3 n^2 \cdot 4^n$$

5. roots are $1, \; 3, 3, 3, \; 6, 6$

$$a_n = \alpha_1 1^n + \alpha_2 3^n + \alpha_3 n \cdot 3^n + \alpha_4 n^2 \cdot 3^n + \alpha_5 6^n + \alpha_6 n 6^n$$