Worth a look in text

5. **Catalan numbers**: Let C_n denote the number of different ways to parenthesize the product of $(n+1)$ numbers, $x_0 \cdot x_1 \cdot x_2 \cdots x_n$.

6. Let w_n represent the number of strings of length n over the alphabet "abcde" with no adjacent e's.

Base cases: $w_1 = 5$, $w_2 = 25 - 1 = 24$, $w_3 = 5^3 - 9 = 116$

Write a recursive formula noting:

- any valid length-$(n-1)$ word can be extended to produce a valid length-n word in one of 4 ways – tack on a, b, c, d.
- any valid length-$(n-2)$ word can be extended to produce a valid length-n word in one of 4 ways – tack on ae, be, ce, de

$$w_n = 4w_{n-1} + 4w_{n-2}$$

Since linear, homog., const. coeff., can use process in 8.2:

$$w_n = r^n \quad \left(w_{n-1} = r^{n-1}, \ w_{n-2} = r^{n-2} \right)$$

Insert

$$r^n = 4r^{n-1} + 4r^{n-2} \quad \text{or} \quad r^{n-2}(r^2 - 4r - 4) = 0$$

$r^2 - 4r - 4 = 0$ has roots

$$r = \frac{4}{2(1)} \pm \frac{1}{2(1)}\sqrt{16 - 4(-4)} = 2 \pm 2\sqrt{2}$$

Now take a weighted sum

$$w_n = \alpha_1 (2 + 2\sqrt{2})^n + \alpha_2 (2 - 2\sqrt{2})^n$$

Now use

$$\begin{cases} 5 = w_1 = \alpha_1 (2 + 2\sqrt{2}) + \alpha_2 (2 - 2\sqrt{2}) \\ 24 = w_2 = \alpha_1 (2 + 2\sqrt{2})^2 + \alpha_2 (2 - 2\sqrt{2})^2 \end{cases}$$

2 eqns lead to finding α_1, α_2.

More recurrences

~~6. Let w_n represent the number of strings of length n over the alphabet "abcde" with no adjacent e's.~~

7. Ways to feed $n to a vending machine with $1 coins, $1 bills, $5 bills

8. Ways to form ternary strings with no consecutive 0s

- first, second, ..., kth differences for a given sequence a_0, a_1, ...

  Find first and second ones for the Fibonacci sequence

For 7, write f_n for the number of ways to feed machine \$n

Symbols: C = \$1 coin, 1 = \$1 bill, 5 = \$5 bill

base:

$f_1 = 2$ — C, 1

$f_2 = 4$ — CC, 11, C1, 1C

$f_3 = 2^3$ — CCC, CC1, C1C, 1CC, C11, 1C1, 11C, 111

$f_4 = 2^4$

$f_5 = 1 + 2^5$

Recursive formula:

$$f_n = 2f_{n-1} + f_{n-5}$$

linear, homog., const. coeff., deg 5

$$r^n = 2r^{n-1} + r^{n-5}$$

becomes $\underbrace{r^5 - 2r^4 - 1 = 0}_{\text{hard to solve}}$

8. Valid strings have

- only 0, 1, 2 as alphabet
- no consecutive 0's.

Denote # of valid length-n strings by w_n.

Note:

- Any length-$(n-1)$ valid string can be turned into a valid length-n string in 2 ways (add 1, 2)
- Any length-$(n-2)$ valid string can be turned into a valid length-n string in 2 ways (add 10, 20)

$$w_n = 2w_{n-1} + 2w_{n-2}$$

linear, homog., const. coeff., degree 2.