Test 3 Wed.

Coverage:

Chapter 5: structural induction

Chapter 8:

8.1 modeling w/ recurrences
none like Exercise 8(a), but given result for that, must be able to things like 8(b) and 8(c)

8.2 solving linear, constant-coefficient, kth degree recurrences
only responsible for homogeneous ones, not (generally) nonhomogeneous, which are treated in the 2nd part of the section.
But see the exception below.

8.3 divide and conquer recursions

Chapter 4: Sections 4.1-4.6

- from 4.2, only need fast modular exponentiation, like in the day's class notes
- know (from class notes) about
  - Euler phi function (how defined, how to evaluate it)
  - Euler's Theorem

Task: Let a_n be the number of ternary strings having 3 consecutive 0s. Find a recurrence relation for a_n.

If you have a valid $(n-1)$-digit ternary string (i.e., one that has 3 consecutive 0's), it will still be valid no matter if we tack on, as a final digit, a 0, 1 or 2. This would account for all valid n-digit strings ending in 1 or 2. However, there are valid n-digit strings ending in 0 that, when considering their first $(n-1)$ digits, are invalid. So, let's only account here for the valid strings ending in 1 or 2, of which there are $2a_{n-1}$, and deal with those ending in 0 separately.

- If it ends in just one zero, as in

  $$\ldots 10 \qquad \text{or} \qquad \ldots 20$$

  then it must have been already valid (already had 3 consecutive zeros) before the final two digits were tacked on. As there are 2 ways to tack on the final two digits so that you end in 0, this contributes $2a_{n-2}$.

- If it ends in exactly two zeros, as in

  $$\ldots 100 \qquad \text{or} \qquad \ldots 200$$

  then it must have been already valid (already had 3 consecutive zeros) before the final three digits were tacked on. As there are 2 ways to tack on the final three digits so that you end in 00, this contributes $2a_{n-3}$.

- If it has 3 or more final zeros, as in

$$\ldots\ldots 0000, \quad \ldots\ldots 1000, \quad \text{or} \quad \ldots\ldots 2000$$

then it doesn't matter whether it was already valid in its first $(n-3)$ digits. Every one of the length $n-3$ ternary strings become a valid length n ternary string when three final zeros are added. So, this group has size 3^{n-3}.

Using the above,

$$\boxed{a_n = 2a_{n-1} + 2a_{n-2} + 2a_{n-3} + 3^{n-3}.}$$

Note: It is this sort of task from Section 8.1 I've said you will not be asked to do on Test 3. But, given this boxed recurrence relation, along with what a_n counts, I might ask for

- initial conditions
- the values of a_4, a_5, a_6, etc.

Homogeneous is an adjective that is not used unless your recurrence is linear.

There are many nonlinear recurrences. One example

$$a_n = 3a_{n-1}a_{n-2} + 5, \quad a_0 = 2, \; a_1 = 3$$

Would not ask you to "solve" (i.e., find closed formula $a_n =$ ______)

Might ask: Find a_2, a_3, a_4

Or, another

$a_n = n a_{n-1}$ linear, but non-constant coeff.

$a_0 = 1$

methods in 8.2 don't work (can't assume $a_n = r^n$.)

$a_1 = 1 \cdot a_0 = 1$

$a_2 = 2 \cdot a_1 = 2 \cdot 1 = 2$

$a_3 = 3 \cdot a_2 = 3 \cdot 2 = 6$

This turns out to be quite solvable: $a_n = n!$

Here is a linear 1st-degree nonhomogeneous recurrence relation:

$$a_n = 3a_n + 2, \quad a_0 = 5.$$

This will not be on Test 3, but we have solved this type before using the "iterative method". (But, be prepared to see it on the final.)

The iterative approach:

$$
\begin{aligned}
a_n &= 3a_{n-1} + 2 \\
&= 3(3a_{n-2} + 2) + 2 = 3^2 a_{n-2} + 3\cdot 2 + 2 \\
&= 3^2(3a_{n-3} + 2) + 3\cdot 2 + 2 = 3^3 a_{n-3} + 3^2\cdot 2 + 3\cdot 2 + 2 \\
&= 3^3(3a_{n-4} + 2) + 3^2\cdot 2 + 3\cdot 2 + 2 = 3^4 a_{n-4} + 3^3\cdot 2 + 3^2\cdot 2 + 3\cdot 2 + 2 \\
&= \cdots \\
&= 3^n a_0 + \underbrace{3^{n-1}\cdot 2 + 3^{n-2}\cdot 2 + \cdots + 3^2\cdot 2 + 3\cdot 2 + 2}_{\text{terms of a geometric sequence, with sum } 2\cdot\frac{3^n-1}{3-1} = 3^n - 1} \\
&= 3^n\cdot a_0 + 3^n - 1 = 3^n\cdot 5 + 3^n - 1 = 3^n(5+1) - 1
\end{aligned}
$$

So

$$\boxed{a_n = 6\cdot 3^n - 1.}$$

Structural induction

On Oct. 22 (see notes), I defined recursively a set S of strings on the alphabet $\{a, b\}$ in this manner:

Base Step: Admit to S the empty string λ and the string aab.

Recursive step: Given words/strings $w_1, w_2, \ldots, w_n \in S$, admit into S the string formed via concatenation $+$:

$$w_1 + w_2 + \cdots + w_n = w_1 w_2 \cdots w_n$$

(So $aab + aab + aabaab + \lambda = aabaabaabaab$.

Claim: Every word in S has length divisible by 3.

(Use structural induction to prove this.)

Basis Step: The two initially-admitted words, λ and aab, have lengths 0 and 3, both being divisible by 3.

Induction Step: Suppose $w_1, w_2, \ldots, w_n \in S$ are to be used to build a new word. The induction hypothesis is that each of $w_1, w_2, \ldots, w_n$ have lengths $\ell(w_i)$ divisible by 3. That is, $\exists$ integers $k_1, k_2, \ldots, k_n$ such that

$$\ell(w_1) = 3k_1, \quad \ell(w_2) = 3k_2, \quad \ldots, \quad \ell(w_n) = 3k_n.$$

The concatenated word $w = w_1 w_2 \cdots w_n$ has length

$$\begin{aligned} \ell(w) &= \ell(w_1) + \ell(w_2) + \cdots + \ell(w_n) \\ &= 3k_1 + 3k_2 + \cdots + 3k_n \\ &= 3\underbrace{(k_1 + k_2 + \cdots + k_n)}_{\text{an integer}}, \end{aligned}$$

showing that $3 \mid \ell(w)$.