## Solutions

1. (b) $\dfrac{3x^4 + 5x^3 + 8x}{2x+1}$ is $\Omega(x^3)$ with witnesses $C = 1,\ k = 1$.

   (d) $\dfrac{3x^4 + 5x^3 + 8x}{2x+1}$ is $O(x^3)$ with witnesses $C = 8,\ k = 1$.

2. Every nonempty subset of $\mathbb{N}$ has a smallest element.

3. (a) They have the same cardinality since they can be put in one-to-one correspondence, the naturals on the left associated with a partner from the other set on the right so that the mapping $n \mapsto 2n$ is bijective:

| $\mathbb{N}$ | | nonnegative evens |
|---|---|---|
| 0 | $\leftrightarrow$ | 0 |
| 1 | $\leftrightarrow$ | 2 |
| 2 | $\leftrightarrow$ | 4 |
| 3 | $\leftrightarrow$ | 6 |
| | $\vdots$ | |
| $n$ | $\leftrightarrow$ | $2n$ |
| | $\vdots$ | |

   (b) $\mathbb{Q}$ and $\mathbb{Z}$ are countably infinite. $\mathbb{R}$ is uncountable. The last set is finite, consisting only of 100 numbers $\{0, 1, 2, \ldots, 99\}$.

4. First, using the formula for the sum of finitely many terms in a geometric series, we have the amount owed after $n$ months is

$$25000r^n - (m + mr + mr^2 + \cdots + mr^{n-1}) \quad = \quad 25000r^n - m\frac{r^n - 1}{r - 1}$$

   We want that amount to be 0 when we plug in $n = 60$, so setting the latter expression equal to 0, we have

$$25000r^{60} - m\frac{r^{60} - 1}{r - 1} = 0 \quad \Rightarrow \quad m\frac{r^{60} - 1}{r - 1} = 25000r^{60}$$
$$\Rightarrow \quad m = 25000r^{60}\frac{r - 1}{r^{60} - 1} = 25000(1.005)^{60}\frac{1.005 - 1}{1.005^{60} - 1} = 483.32.$$

5. We have

$$\begin{aligned} a_n &= a_{n-1} + 13 = (a_{n-2} + 13) + 13 \\ &= a_{n-2} + (13)(2) = (a_{n-3} + 13) + 2(13) \\ &= a_{n-3} + (13)(3) = (a_{n-4} + 13) + 3(13) \\ &= a_{n-4} + (13)(4) = \cdots \\ &= a_0 + 13n = 11 + 13n. \end{aligned}$$

7. (a) It is not true for $n = 1$, but is true for $n = 2, 3$, and so on. So, it seems the set $A = \{2, 3, 4, \ldots\}$.

   (b) **Basis step**, $S(2)$:
   The left side, $2^2 = 4$, is less than the right side $(2 + 1)! = 6$.

   **Induction step**, $S(k) \rightarrow S(k + 1)$:
   We assume that, for some $k \geq 2$, $S(k)$ holds, namely, $2^k < (k + 1)!$. To prove $S(k + 1)$, we start with

the quantity $2^{k+1}$. In what follows, the induction hypothesis is used at the first instance of an inequality.

$$2^{k+1} = 2^k \cdot 2 < (k+1)!(2) < (k+1)!(k+2) = (k+2)!.$$

8. (a) The base case is $P(1)$. When both piles have just 1 matchstick, Player 1 is forced to take one emptying one of the piles. Then Player 2 wins by taking the 1 from the other pile.

   (b) His proof uses strong induction. There is, after all, the possibility that $1 < j < k+1$. So after Player 1 removes $j$ matchsticks (more than one, but not all) from one pile and Player 2 mirrors that action, drawing $j$ from the other, we rely on $P(k+1-j)$ to be true. In mathematical induction, our induction hypothesis would only allow us to assume $P(k)$ was true, which only helps if $j = 1$. In strong induction we get to assume more cases.

   (c) Our induction hypothesis is that for some integer $k \geq 1$, the compound proposition

   $$P(1) \wedge P(2) \wedge \cdots \wedge P(k)$$

   holds true. That is, at any initial number of matchsticks $j$ with $1 \leq j \leq k$ in both piles, Player 2 can guarantee a win.

   (d) The set $A = \mathbb{Z}^+$, the positive integers.