1. (a)

$$306 = 84 \cdot \underline{3} + \underline{54}$$
$$84 = 54 \cdot \underline{1} + \underline{30}$$
$$54 = 30 \cdot \underline{1} + \underline{24}$$
$$30 = 24 \cdot \underline{1} + \underline{6}$$
$$24 = 6 \cdot \underline{4} + \underline{0}$$

last nonzero remainder is gcd(306, 84).

(b)

$$51 = 14 \cdot \underline{3} + \underline{9} \quad \text{or} \quad r_2 = 9 = 51 - (14)(3) = r_0 - 3r_1$$
$$14 = 9 \cdot \underline{1} + \underline{5} \quad \text{or} \quad r_3 = 5 = 14 - 9 = r_1 - r_2$$
$$9 = 5 \cdot \underline{1} + \underline{4} \quad \text{or} \quad r_4 = 4 = 9 - 5 = r_2 - r_3$$
$$5 = 4 \cdot \underline{1} + \underline{1} \quad \text{or} \quad \gcd(51, 14) = 1 = 5 - 4 = r_3 - r_4$$

To reassemble, we have

$$\gcd(51,14) = r_3 - r_4 = r_3 - (r_2 - r_3) = 2r_3 - r_2$$
$$= 2(r_1 - r_2) - r_2 = 2r_1 - 3r_2 = 2r_1 - 3(r_0 - 3r_1)$$
$$= 11r_1 - 3r_0, \quad \text{or} \quad 1 = 51s + 14t \quad \text{with } s = -3, t = 11.$$

(c) Each of 84, 54, and 306 is divisible by 6. So

$$84x \equiv 54 \pmod{306} \quad \text{gives way to} \quad 14x \equiv 9 \pmod{51}.$$

From part (b), we deduce 11 is the multiplicative inverse of 14 in $\mathbb{Z}_{51}$.

So,
$$x \equiv (11)(14x) \equiv (11)(9) \equiv 48 \pmod{51}.$$

In the integers

$$48, 99, 150, 201, 252, 303, 354, 405, \ldots$$

are all equivalent mod 51. Back in mod 306, there is no redundancy until you reach 354. Thus

$$\boxed{x \equiv 48, 99, 150, 201, 252, 303 \pmod{306}}$$

2. $49 = 7^2$, $93 = 31 \cdot 3$, $119 = 17 \cdot 7$, so these three are not prime.

But no prime number less than $\sqrt{67}$ divides 67
no prime number less than $\sqrt{71}$ divides 71
no prime number less than $\sqrt{193}$ divides 193

So these three are primes

3. (a) 71 is prime (see above), and $71 \nmid 68$. Thus, by Fermat's Little Theorem, $68^{70} \equiv 1 \pmod{71}$, and $68^{71} \equiv 68 \pmod{71}$

So, $68^{712} = 68^{710} \cdot 68^2 = (68^{71})^{10} 68^2 \equiv (68)^{10} \cdot 68^2$

$\equiv (-3)^{12} \equiv (3^4)^3 = 81^3 \equiv 10^3 = 1000 \equiv 6 \pmod{71}$.

This means $68^{712} \bmod 71 = \boxed{6}$

(b) $49 = 7^2$ and $85 = 5 \cdot 17$, so $\gcd(49, 85) = 1$, meaning that Euler's Thm applies. $\varphi(85) = \varphi(5)\varphi(17) = (4)(16) = 64$.

So $49^{64} \equiv 1 \pmod{85}$. Hence

$49^{642} = 49^{640} \cdot 49^2 = (49^{64})^{10} \cdot 49^2 \equiv 1^{10} \cdot 49^2$

$= 7^3 \cdot 7 = 343 \cdot 7 \equiv 3 \cdot 7 \equiv 21 \pmod{85}$.

Hence $49^{642} \bmod 85 = \boxed{21}$

4. (a) $\varphi(42) = \varphi(7)\varphi(3)\varphi(2) = 6 \cdot 2 \cdot 1 = \boxed{12}$

(b) $\varphi(45^3) = \varphi(3^6 \cdot 5^3) = 3^6 \cdot 5^3 \cdot (1 - \frac{1}{3})(1 - \frac{1}{5}) = 48600$

5. (a) This recurrence is 2^{nd} degree, and only requires two initial values.

$a_0 = 0, \quad a_1 = 0.$

(b) $a_2 = 2a_1 + a_0 + 2 \cdot 3^0 = 2$

$a_3 = 2a_2 + a_1 + 2 \cdot 3^1 = (2)(2) + 6 = 10$

$a_4 = 2a_3 + a_2 + 2 \cdot 3^2 = (2)(10) + 2 + 18 = 40$

$a_5 = 2a_4 + a_3 + 2 \cdot 3^3 = (2)(40) + 10 + 54 = 144$

$a_6 = 2a_5 + a_4 + 2 \cdot 3^4 = (2)(144) + 40 + 162 = \boxed{490}$

6. Substituting r^n for a_n (r^{n-1} for a_{n-1}, r^{n-2} for a_{n-2}) takes us from

$$a_n - 7a_{n-1} + 10a_{n-2} = 0 \quad \text{to} \quad r^2 - 7r + 10 = 0$$

or $$(r-5)(r-2) = 0 \implies r = 2, 5.$$

Terms in the sequence are a weighted sum of 2^n and 5^n:

$$a_n = \alpha_1 \cdot 2^n + \alpha_2 \cdot 5^n.$$

Our initial values give

$$2 = a_0 = \alpha_1 \cdot 2^0 + \alpha_2 \cdot 5^0 = \alpha_1 + \alpha_2$$

$$1 = a_1 = \alpha_1 \cdot 2^1 + \alpha_2 \cdot 5^1 = 2\alpha_1 + 5\alpha_2$$

These 2 equations in unknown weights α_1, α_2

$$\left.\begin{aligned} \alpha_1 + \alpha_2 &= 2 \\ 2\alpha_1 + 5\alpha_2 &= 1 \end{aligned}\right\} \text{ can be solved to obtain } \begin{aligned} \alpha_1 &= 3 \\ \alpha_2 &= -1 \end{aligned}$$

So, the solution is $$\boxed{a_n = 3 \cdot 2^n - 5^n}$$

7. (a) Here, $a = 3$, $b = 2$, $c = 1$ and $d = 2$, which means $a < b^d$. Thus,

$T(n)$ is $O(n^2)$.

(b) This time, $a = 4$, $b = 2$, $c = \frac{1}{2}$, $d = 1$, so $a > b^d$. Thus,

$g(n)$ is $O(n^{\log_2 4}) = O(n^2)$ again.

8. Base step: The sum of digits in λ is 0 and $3 | 0$

The sum of digits in "27" is 9 and $3 | 9$

The sum of digits in "9" is 9 and $3 | 9$

The sum of digits in "414" is 9 and $3 | 9$

Induction step: Our new word $w = w_1 w_2$ is the concatenation of words w_1, w_2 and, by the induction hypothesis, w_1 and w_2 satisfy the claim. Since the sum of digits in w is merely the sum of digits in w_1 plus the sum of digits in w_2, and since the sum of any two numbers both divisible by 3 is again divisible by 3, it follows that w also satisfies the claim.